$T_2(\vec{x}) = A_2\vec{x}$ where $A_2 = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & 1 \end{bmatrix}$; $\det(A_2) = -12$.

$\therefore$ T_2 is bijective.

Several linear transformations are stated on below and on the next page. For each transformation, determine if the transformation is bijective, only injective, only surjective, or nojective. Then:

- If the transformation is not injective, determine an explicit, non-zero vector $\vec{x}$ that maps to $\vec{0}$. Show work that leads to the vector. That work might simply be a statement that begins "(i)t is obvious that …". Finally, use the stated pattern to explicitly show that $\vec{x}$ maps to $\vec{0}$.
- If the transformation is not surjective, determine an explicit vector in the codomain that is not in the range of the transformation. Show work that leads to the vector. That work might simply be a statement that begins "(i)t is obvious that …".

Linear Transformation 1: $T_1\left(\left[x_1, x_2, x_3\right]^T\right) = \left[-2x_1 + x_3, x_1 + x_2, 3x_1 + x_2 - x_3\right]^T$

Linear Transformation 2: $T_2\left(\left[x_1, x_2, x_3\right]^T\right) = \left[3x_2, -x_3, 4x_1 + x_3\right]^T$

$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + x_3 \\ x_1 + x_2 \\ 3x_1 + x_2 - x_3 \end{bmatrix}$; $T_1(\vec{e}_1) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$, $T_1(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $T_1(\vec{e}_3) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\therefore$ $T_1(\vec{x}) = A_1\vec{x}$ where $A_1 = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$

$T: \mathbb{R}^3 \to \mathbb{R}^3$; $\det(A_1) = 0$ $\therefore$ T_1 is nojective

codomain

Prove not injective: Find some non-zero vector that gets mapped to the zero vector

$\begin{bmatrix} -2x_1 + x_3 \\ x_1 + x_2 \\ 3x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Clearly $x_1 = \frac{1}{2}x_3$ & $x_2 = -\frac{1}{2}x_3$, so a solution is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

$T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2(1) + 2 \\ 1 + (-1) \\ 3(1) + (-1) - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

Prove not surjective

$\begin{bmatrix} -2x_1 + x_3 \\ x_1 + x_2 \\ 3x_1 + x_2 - x_3 \end{bmatrix}$ $x_1 = 1, x_3 = 1 \to$ / $x_1 = 1, x_2 = 1 \to$ / $x_1 = 1, x_2 = 1, x_3 \neq 1$ & $x_3 = 2$ $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$; $\left[\begin{array}{ccc|c} -2 & 0 & 1 & -1 \\ 1 & 1 & 0 & 2 \\ 3 & 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

contradiction QED

Linear Transformation 3: $T_3\left(\left[x_1, x_2\right]^{\mathrm{T}}\right)=\left[x_2, 0, x_1-x_2\right]^{\mathrm{T}}$

Linear Transformation 4: $T_4\left(\left[x_1, x_2, x_3, x_4\right]^{\mathrm{T}}\right)=\left[2x_1+x_2-x_4, x_1+x_2-x_3+2x_4, -x_2+2x_3-5x_4\right]$

$T_3 : \mathbb{R}^2 \to \mathbb{R}^3$ (domain, codomain) T_3 cannot be surjective (or bijective)

$T_3(\vec{x}) = A_3 \vec{x}$ where $A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$ A_3 is not square, so there's no determinant and no Theorem 8. Dang!

Test for injectiveness: is there a nonzero vector that maps to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \\ x_1 - x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ 0 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By observation $x_2 = 0$ (row 1)

So, by observation, $x_1 = x_2 = 0$ (row 3). $\therefore T_3$ is only injective.

$T_4(\vec{x}) = \begin{bmatrix} 2x_1 + x_2 - x_4 \\ x_1 + x_2 - x_3 + 2x_4 \\ -x_2 + 2x_3 - 5x_4 \end{bmatrix}$ $T: \mathbb{R}^4 \to \mathbb{R}^3$ so it cannot be injective (or bijective)

$T_4(\vec{x}) = A_4 \vec{x}$ where $A_4 = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -5 \end{bmatrix}$ nay, almost every

$\left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & a \\ 1 & 1 & -1 & 2 & b \\ 0 & -1 & 2 & -5 & c \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -3 & * \\ 0 & 1 & -2 & 5 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$ ← clearly, ~~for~~ ~~some~~ $\vec{v} \in \mathbb{R}^3$ this bottom row is going to be a contradiction.

$\begin{bmatrix} 2x_1 + x_2 - x_4 \\ x_1 + x_2 - x_3 + 2x_4 \\ -x_2 + 2x_3 - 5x_4 \end{bmatrix}$ $x_1 = 1, x_2 = 1, x_4 = 1$; $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$; $\underbrace{x_2 \neq 1}_{x_2 = 0}, x_3 = 1, x_4 = 1$ $\begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$

$\left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 2 \\ 1 & 1 & -1 & 2 & 3 \\ 0 & -1 & 2 & -5 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$\therefore \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} \notin \text{range}(T_4)$ so T_4 is not surjective.

$\therefore T_4$ is "no"jective