Matrices and vectors – a few introductory definitions

An $m \times n$ ***matrix*** is a rectangular array of numbers with m rows and n columns. m and n are called the ***dimensions*** of the matrix.

Matrices are most commonly denoted by capitol letters or single subscripted capitol letters. The numbers (***entries***) of a matrix are denoted by double subscripted lower case letters.

When a matrix is explicitly written out, it is delineated by square brackets or parentheses. Abstractly, you will see things like $A = \left[\, a_{ij} \,\right]$ where a_{ij} represents the entry in the i^{th} row and j^{th} column. When the number of rows or columns goes above 9, we separate the *i*s and *j*s by commas; we won't be going there.

Example

Consider the matrix $B = \begin{bmatrix} 4 & 7 & -7 \\ 8 & -2 & 0 \\ -4 & 11 & 9 \\ -18 & -62 & 3 \\ 22 & -1 & 14 \end{bmatrix}$.

a. What are the dimensions of the matrix?

5×3

b. What are entries b_{32}, b_{23}, b_{41}, and b_{14}?

$b_{32} = 11$

$b_{23} = 0$

$b_{41} = -18$

There ain't no b_{14}.

A matrix with only one row is called a ***row vector*** and a matrix with only one column is called a ***column vector***.

Example

Which of the following are vectors?

i. $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 7 \end{bmatrix}$ ii. $\begin{bmatrix} 3 & -1 \\ 6 & 12 \end{bmatrix}$ iii. $[4, 7, -18]$ iv. $[5 \quad 0]$ v. 

Matrices – a couple of simple arithmetic operations

Matrix addition: Two matrices *with the exact same dimensions* can be added or subtracted thusly:

$$\left[a_{ij}\right] \pm \left[b_{ij}\right] = \left[a_{ij} \pm b_{ij}\right]$$

If either of the corresponding dimensions of two matrices is not the same, the matrices can neither be added nor subtracted.

Scalar multiplication: A matrix can be multiplied by a ***scalar*** (number) thusly:

$$k\left[a_{ij}\right] = \left[k\, a_{ij}\right]$$

Example

Consider the matrices $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \\ 0 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 8 & -2 \\ -5 & -1 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & -2 & 6 \\ 3 & -2 & -2 \end{bmatrix}$.

Find $A + B$, $B - 2C$, and $C + A^t$

> The ***transpose*** of the $m \times n$ matrix M is the $n \times m$ matrix, M^t, that results from swapping the rows and columns of M.

A+B cannot be done; the dimensions are mismatched

$$B - 2C = \begin{bmatrix} 4 & 8 & -2 \\ -5 & -1 & 6 \end{bmatrix} - 2\begin{bmatrix} -1 & -2 & 6 \\ 3 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-(-2) & 8-(-4) & -2-12 \\ -5-6 & -1-(-4) & 6-(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 & -14 \\ -11 & 3 & 10 \end{bmatrix}$$

$$C + A^t = \begin{bmatrix} -1 & -2 & 6 \\ 3 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 0 & -9 \end{bmatrix}$$

Which of the following are column vectors?

i. $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$ ii. $\begin{bmatrix} 3 & -1 \\ 6 & 12 \end{bmatrix}$ iii. $[4, 7, -18]^t$ iv. $[5 \quad 0]$ v.

Generalized Matrix Multiplication

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the product AB is the $m \times n$ matrix whose ij^{th} entry is the dot product of the i^{th} row of A and the j^{th} column of B

If A is an $m \times p$ matrix and B does not have p rows, then the product AB is not defined.

Examples

Find AB if $A = \begin{bmatrix} 2 & 1 \\ 4 & 6 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 0 & 7 \\ -3 & 5 & -4 & -5 \end{bmatrix}$.

What are the dimensions of the product?

$3 \times 2 \qquad 2 \times 4$

Good to go

Product is 3×4

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 & 7 \\ -3 & 5 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(-1)+(1)(-3) & (2)(3)+(1)(5) & 0+(-4) & 14+(-5) \\ (4)(-1)+(6)(-3) & (4)(3)+(6)(5) & 0+(-24) & 28+(-30) \\ (3)(-1)+(-2)(-3) & (3)(3)+(-2)(5) & 0+8 & 21+10 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 11 & -4 & 9 \\ -22 & 42 & -24 & -2 \\ 3 & -1 & 8 & 31 \end{bmatrix}$$

Find AB if $A=\begin{bmatrix} 2 & 1 & 0 & 8 & -5 \\ 1 & 3 & 4 & -3 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -2 \\ 10 & 1 \\ 2 & 4 \\ 0 & 3 \\ -6 & 7 \end{bmatrix}$.

What are the dimensions of the product?

$\underbrace{2\times 5 \quad 5}\times 2$

Good

product is 2×2

$$AB = \begin{bmatrix} 2 & 1 & 0 & 8 & -5 \\ 1 & 3 & 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 10 & 1 \\ 2 & 4 \\ 0 & 3 \\ -6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 6+10+0+0+30 & -4+1+0+24-35 \\ 3+30+8+0+6 & -2+3+16-9-7 \end{bmatrix}$$

$$= \begin{bmatrix} 46 & -14 \\ 47 & 1 \end{bmatrix}$$

Find AB if $A=\begin{bmatrix} 2 & 1 & 0 & 8 & -5 \\ 1 & 3 & 4 & -3 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -2 \\ 10 & 1 \\ 2 & 4 \end{bmatrix}$.

What are the dimensions of the product?

$\underbrace{2\times 5 \quad 3}\times 2$

mismatch

AB cannot be performed

Write a simple dimensional proof that establishes that in general, matrix multiplication is not commutative; i.e., in general $AB \neq BA$.

If A is 2×5 and B is 3×2,

BA exists but AB does not.

$\therefore$ In general $AB \neq BA$

If C is 2×5 and D is 5×2

CD is 2×2 whereas $DC = 5\times 5$.

Inverses are unique:

Suppose that $AB = AC = I$.

Then: $A^{-1}AB = A^{-1}AC = A^{-1}I$

$\Rightarrow IB = IC = A^{-1}$

$\Rightarrow B = C = A^{-1}$

Use your calculator to find I_3A and AI_3 where $A = \begin{bmatrix} 2 & 6 & -3 \\ -1 & 5 & 4 \\ 9 & 12 & -7 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_3A = A$

$AI_3 = \begin{bmatrix} 2 & 6 & -3 \\ -1 & 5 & 4 \\ 9 & 12 & -7 \end{bmatrix} = A$

Use your calculator to find AB where $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}$.

$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

A and B are <u>inverse matrices</u>. $A = B^{-1}$; $B = A^{-1}$

$AB = I$

Some definitions

The $n \times n$ matrix with entries of 1 along the main diagonal and 0 in every other slot is called the $n \times n$ ***identity matrix*** and is denoted as I_n.

If the products are defined, $AI_n = A$ and/or $I_nA = A$ for any compatible matrix A.

A **square** matrix A with dimensions $n \times n$ is called ***invertible*** if there exists an $n \times n$ matrix B with the property $AB = I_n$. If such a matrix exists we call it ***the inverse of A*** and symbolize it as A^{-1}.

If the $n \times n$ A is invertible, then $AA^{-1} = A^{-1}A = I_n$.

We do not define inverse matrices for non-square matrices.

Examples

Which of the following is $\mathbf{C}^{-1}$, where $\mathbf{C} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$?

a. $\mathbf{A} = \begin{bmatrix} 1/5 & 1/3 \\ 1/2 & 1 \end{bmatrix}$ b. $\mathbf{B} = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$ c.  d. $\mathbf{D} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$

$CB = \begin{bmatrix} -25 & -6 & \sim \\ \sim & & \sim \end{bmatrix}$ No Good

$CD = \begin{bmatrix} -5+6 & 15-15 \\ -2+2 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A game plan for determining, **by hand**, the inverse of the matrix $A=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$ is shown below.

Let's execute the plan on the next page.

Let $A^{-1}=\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}$. We need $AA^{-1}=I_3$ which gives us:

$$\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}b_{11}-b_{21} & b_{12}-b_{22} & b_{13}-b_{23}\\ b_{11}-b_{31} & b_{12}-b_{32} & b_{13}-b_{33}\\ 6b_{11}-2b_{21}-3b_{31} & 6b_{12}-2b_{22}-3b_{32} & 6b_{13}-2b_{23}-3b_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

If we equate the columns of the respective matrices, we come up with three systems of three equations with three unknowns. Specifically:

$$\begin{cases}b_{11}-b_{21}=1\\ b_{11}-b_{31}=0\\ 6b_{11}-2b_{21}-3b_{31}=0\end{cases},\quad \begin{cases}b_{12}-b_{22}=0\\ b_{12}-b_{32}=1\\ 6b_{12}-2b_{22}-3b_{32}=0\end{cases},\ \text{and}\ \begin{cases}b_{13}-b_{23}=0\\ b_{13}-b_{33}=0\\ 6b_{13}-2b_{23}-3b_{33}=1\end{cases}$$

A not-so-close inspection should convince you that not only are the coefficient matrices for all three of the systems identical, but they are all in fact the matrix A! Since the row operations performed in Gaussian elimination are determined solely by the coefficient matrix, we may as well go ahead and solve all three systems simultaneously. The augmented matrix representation for these three systems is:

$$\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ 6 & -2 & -3 & 0 & 0 & 1\end{array}\right]$$

common coefficient matrix

Constant terms, from left to right, for the first, second, and third columns' related system of equations.

Let $A = \begin{bmatrix} -12 & -5 & -9 \\ -4 & -2 & -4 \\ -8 & -4 & -6 \end{bmatrix}$. Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} -12 & -5 & -9 & 1 & 0 & 0 \\ -4 & -2 & -4 & 0 & 1 & 0 \\ -8 & -4 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} -4 & -2 & -4 & 0 & 1 & 0 \\ -12 & -5 & -9 & 1 & 0 & 0 \\ -8 & -4 & -6 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} -4 & -2 & -4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} 2R_3 + R_1 \to R_1 \\ -\frac{3}{2}R_3 + R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} -4 & -2 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right]$$

$$2R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 2 & -3 & -1 \\ 0 & 1 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} -\frac{1}{4}R_1 \to R_1 \\ \frac{1}{2}R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3/4 & 1/4 \\ 0 & 1 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -1 & 1/2 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1/2 & 3/4 & 1/4 \\ 1 & 0 & -3/2 \\ 0 & -1 & 1/2 \end{bmatrix}$$

Check: $\begin{bmatrix} -12 & -5 & -9 \\ -4 & -2 & -4 \\ -8 & -4 & -6 \end{bmatrix} \begin{bmatrix} -1/2 & 3/4 & 1/4 \\ 1 & 0 & -3/2 \\ 0 & -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Phew!

$\therefore$

Find – **by hand** - if they exist, the inverses of each of the following matrices.

$$A=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix} \text{ and } B=\begin{bmatrix}7 & -12\\ 18 & 4\\ 0 & -9\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ 6 & -2 & -3 & 0 & 0 & 1\end{array}\right] \begin{array}{l}-R_1+R_2\to R_2\\ -6R_1+R_3\to R_3\end{array} \left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 4 & -3 & -6 & 0 & 1\end{array}\right]$$

$$-4R_2+R_3\to R_3 \left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

$$R_3+R_2\to R_2 \left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -3 & -3 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

$$R_2+R_1\to R_1 \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & -3 & 1\\ 0 & 1 & 0 & -3 & -3 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

$$\therefore\ A^{-1}=\begin{bmatrix}-2 & -3 & 1\\ -3 & -3 & 1\\ -2 & -4 & 1\end{bmatrix}$$

check: $\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}\begin{bmatrix}-2 & -3 & 1\\ -3 & -3 & 1\\ -2 & -4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

B is not a square matrix so it has no inverse.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

All of this only applies to 2x2 matrices

Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{-\frac{c}{a}R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\frac{a}{ad-bc}R_2 \to R_2 \quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$-bR_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|cc} a & 0 & \frac{bc}{ad-bc} + 1 & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\frac{1}{a}R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Let's us the formula we just derived to help us solve the system $\begin{cases} 4x_1 + 3x_2 = 20 \\ -2x_1 + 5x_2 = -36 \end{cases}$.

This system can be written:

$$\begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ -36 \end{bmatrix}$$

So....

$$\begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 20 \\ -36 \end{bmatrix}$$

which gives $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 20 \\ -36 \end{bmatrix}$

The upshot is: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 20 \\ -36 \end{bmatrix}$

$$= \frac{1}{20-(-6)}\begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 20 \\ -36 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 208 \\ -104 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Let $A=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix}$. Find, by hand, AB, BA, $(AB)^T$, $(BA)^T$, A^TB^T, B^TA^T, A^{-1}, B^{-1}, $(A^{-1})^{-1}$, $(AB)^{-1}$, $(BA)^{-1}$, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$, $(A^{-1})^T$, and $(A^T)^{-1}$. See what equals what.

$$AB = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix} = \begin{bmatrix}13 & 4\\ 23 & 7\end{bmatrix}$$

$$BA = \begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} = \begin{bmatrix}-11 & -17\\ 20 & 31\end{bmatrix}$$

$$(AB)^T = \begin{bmatrix}13 & 23\\ 4 & 7\end{bmatrix}$$

$$(BA)^T = \begin{bmatrix}-11 & 20\\ -17 & 31\end{bmatrix}$$

$$B^TA^T = \begin{bmatrix}-4 & 7\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} = \begin{bmatrix}13 & 23\\ 4 & 7\end{bmatrix}$$

$$A^TB^T = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}-4 & 7\\ -1 & 2\end{bmatrix} = \begin{bmatrix}-11 & 20\\ -17 & 31\end{bmatrix}$$

$$A^{-1} = \frac{1}{10-9}\begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix} = \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}$$

$$B^{-1} = \frac{1}{-8+7}\begin{bmatrix}2 & 1\\ -7 & -4\end{bmatrix} = \begin{bmatrix}-2 & -1\\ 7 & 4\end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{10-9}\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{91-92}\begin{bmatrix}7 & -4\\ -23 & 13\end{bmatrix} = \begin{bmatrix}-7 & 4\\ 23 & -13\end{bmatrix}$$

$$(BA)^{-1} = \frac{1}{-341+340}\begin{bmatrix}31 & 17\\ -20 & -11\end{bmatrix} = \begin{bmatrix}-31 & -17\\ 20 & 11\end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix}-2 & -1\\ 7 & 4\end{bmatrix}\begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix} = \begin{bmatrix}-7 & 4\\ 23 & -13\end{bmatrix}$$

$$A^{-1}B^{-1} = \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2 & -1\\ 7 & 4\end{bmatrix} = \begin{bmatrix}-31 & -17\\ 20 & 11\end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix} \qquad (A^T)^{-1} \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}$$

Conclusion: $(AB)^T = B^TA^T$, $(BA)^T = A^TB^T$, $(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1}$, $(BA)^{-1} = A^{-1}B^{-1}$, $(A^{-1})^T = (A^T)^{-1}$

All of the above are always true for compatible invertible matrices.

We also saw that $A^T = A$. This cannot be true for all matrices because $B^T \neq B$. Matrices that equal their transpose are called <u>Symmetric</u> matrices.