A ***column vector*** is a matrix with only one column. Assuming that we limit the entries to real numbers, the set of all 2 x 1 (column) vectors is called $\mathbb{R}^2$ and the set of all 3 x 1 vectors is called $\mathbb{R}^3$.

Scalar multiplication is the process of multiplying a vector by a real number; the process is affected by multiplying each entry in the vector by the scalar.

Vector addition and ***vector subtraction*** are affected by adding or subtracting the corresponding entries of the two vectors; both of these operations can only be performed between vectors with the same number of rows.

Example

Simplify $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$ and illustrate the process on Figure 1.

$$-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}(-2)(-1)\\(-2)(-2)\end{bmatrix}+\begin{bmatrix}(3)(0)\\(3)(-2)\end{bmatrix}$$

$$= \begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}0\\-6\end{bmatrix}$$

$$= \begin{bmatrix}2+0\\4+(-6)\end{bmatrix}$$

$$= \begin{bmatrix}2\\-2\end{bmatrix}$$

resultant $= \begin{bmatrix}2\\-2\end{bmatrix}$ ✓

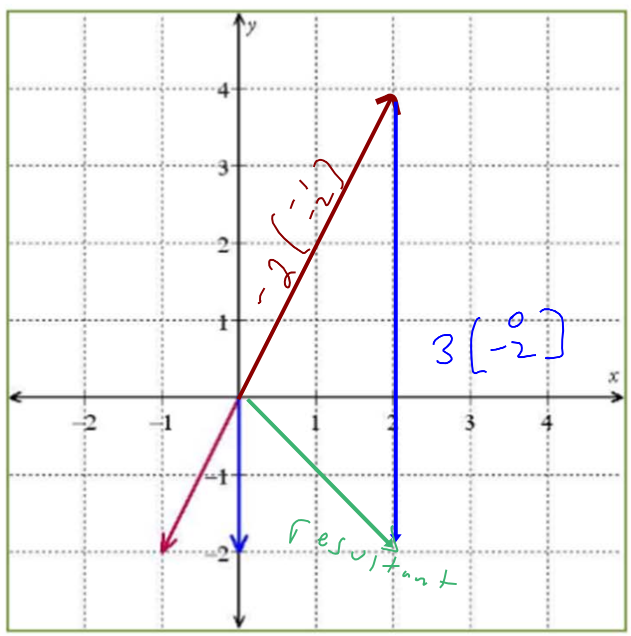


Figure 1: $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$

flips arrow 180°

stretch by a factor of 3

stretches the arrow by a factor of 2

Top 10 definition.

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$. We say that $\vec{y}$ is a ***linear combination*** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ if and only if there exist scalars, $c_1, c_2, \dots, c_p$ such that $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_p\vec{v}_p$. If such scalars exist, they are called the ***weights*** in the linear combination.

Example

Let $\vec{a}_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$. Express $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

Translation:

Solve $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$

for x_1 & x_2

$$x_1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + (-5x_2) \\ -4x_1 + (-x_2) \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 - 5x_2 \\ -4x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 - 5x_2 = 12 \\ -4x_1 - x_2 = -13 \end{cases}$$

$\vec{a}_1$ $\quad \vec{a}_2$ $\quad \vec{b}$

$$\left[\begin{array}{cc|c} 2 & -5 & 12 \\ -4 & -1 & -13 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & -1 \end{array}\right]$$

The columns are $\vec{a}_1$, $\vec{a}_2$, & $\vec{b}$

Check

$$\frac{7}{2}\begin{bmatrix} 2 \\ -4 \end{bmatrix} + (-1)\begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix} \checkmark$$

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ is called the ***span*** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

Top 10 vocabulary word

The vector $\begin{bmatrix} 3 \\ -28/11 \\ 2 \end{bmatrix}$ is in the column space of A which is two dimensional (4 weeks ahead)

Example

Let $A = \begin{bmatrix} 3 & -1 & 5 \\ -2 & 0 & -4 \\ -1 & 4 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$. Find the value of h if $\vec{b}$ is in the span of the columns of A.

$\vec{b}$ in the span of $\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$

translate: $\vec{b}$ is a linear combination of $\vec{c}_1, \vec{c}_2, \vec{c}_3$.

translate: $x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 = \vec{b}$ has at least one solution.

finally, an actionable task!

Real Starting point: $x_1\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$ begets

$$\begin{cases} 3x_1 - x_2 + 5x_3 = 3 \\ -2x_1 \qquad - 4x_3 = h \\ -x_1 + 4x_2 + 2x_3 = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & h \\ -1 & 4 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ -2 & 0 & -4 & h \\ 3 & -1 & 5 & 3 \end{array}\right]$$

$$\begin{array}{l} -2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & -4+h \\ 0 & 11 & 11 & 9 \end{array}\right]$$

$$\frac{11}{8}R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & -4+h \\ 0 & 0 & 0 & \frac{7}{2} + \frac{11}{8}h \end{array}\right]$$

For the attendant system to have solution, we need $\frac{7}{2} + \frac{11}{8}h = 0$

$\therefore h = -28/11$

Check: $\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & -28/11 \\ -1 & 4 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 14/11 \\ 0 & 1 & 1 & 9/11 \\ 0 & 0 & 0 & 0 \end{array}\right]$

WRAP THE CONCEPT UP.

general solution: $\begin{cases} x_1 = -2x_3 + 14/11 \\ x_2 = -x_3 + 9/11 \\ x_3 \text{ is free} \end{cases}$

So a specific solution (letting $x_3 = 9/11$) is $\begin{bmatrix} -4/11 \\ 0 \\ 9/11 \end{bmatrix}$

That is $\begin{bmatrix} 3 \\ -28/11 \\ 2 \end{bmatrix} = (-4/11)\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + (9/11)\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$ ✓

$M_1 d_1 = M_2 d_2$ (balance achieved!)

Application: Center of Mass

The center of mass (or center of gravity) of an object is the average of the product of mass points in the object with their relative distance from a fixed reference point. The concept is most easily understood if on thinks about a teeter-totter. When two people of different mass sit on a teeter totter of uniform density, the center of mass is clearly going to be closer to the heavier person than the lighter person; this phenomenon is reflected in the fact that if the teeter-totter is to stay in balance, the heavier person needs to sit closer to the axis-point than the lighted person. Unfortunately, we can't define the center of mass in all cases as the balance point, because the center of mass is frequently not even on the object! This is easy to see if you think about washer (as in nuts and bolts) of uniform density; the center of mass is clearly the center point of the hole in the middle of the washer.

For most objects, it takes a double or triple integral to calculate the center of mass. In some simple situations, however, the point can be determined by a simple formula. For example, if we have a triangular lamina referenced to the xy-plane, the center of mass ($\vec{v}$) can be determined using the formula $\vec{v} = \frac{1}{3}[\vec{v}_1 + \vec{v}_2 + \vec{v}_3]$ where $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ represent the vectors from the origin to the three vertices of the triangle; for ease of reference, we shall contextually refer to these vectors as points.

That formula is a special case of the more general formula for the center of mass of several mass points.

The center of mass ($\vec{v}$) of n mass points is: $\vec{v} = \frac{1}{m}\sum_{i=1}^{n}[m_i v_i]$ where m_i is the mass at point v_i and m is the sum of all the masses.

Example

Find the center of mass of the triangular lamina outlined in Figure 2 assuming that the lamina has uniform density.

Center of mass is found by assuming that there is a 1 gm weight at each of A, B, and C

$$\vec{v} = \underbrace{\frac{1}{1+1+1}}_{\text{total mass}} \cdot \left[1\begin{bmatrix} -3 \\ -4 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 5 \\ -4 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

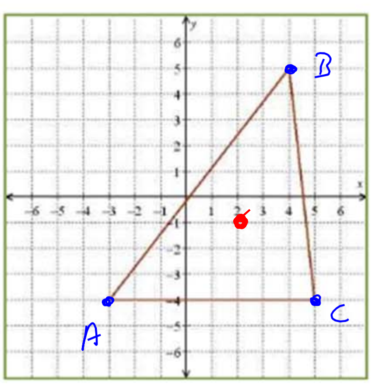


Figure 2: lamina of uniform density and thickness

Example

In the last example we found the center of mass using the formula $\vec{v} = \frac{1}{m}\sum_{i=1}^{n}[m_i v_i]$ and assuming that there was 1 g of mass at each of the vertices. Suppose that we had 9 additional grams of mass to distribute among the vertices. How should the mass be distributed so that the center of mass of the lamina shifts to the point $(3,-2)$?

Assume that there is already one gram at each pt.

Define m_A, m_B, & m_C to be the net weight at each point.

$$\vec{v} = \frac{1}{12}\left[m_A A + m_B B + m_C C\right]$$

$\vec{v}$ desired center of mass

$$\frac{1}{12}\left(m_A\begin{bmatrix}-3\\-4\end{bmatrix} + m_B\begin{bmatrix}4\\5\end{bmatrix} + m_C\begin{bmatrix}5\\-4\end{bmatrix}\right) = \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{cases} -3m_A + 4m_B + 5m_C = 36 \\ -4m_A + 5m_B - 4m_C = -24 \end{cases}$$

uh oh! We need a third equation to get a unique solution. Behold: $m_a + m_b + m_c = 12$ Phew!

$$\left[\begin{array}{ccc|c} -3 & 4 & 5 & 36 \\ -4 & 5 & -4 & -24 \\ 1 & 1 & 1 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/3 \\ 0 & 1 & 0 & 8/3 \\ 0 & 0 & 1 & 20/3 \end{array}\right]$$

To move the center of mass to $\begin{bmatrix}3\\-2\end{bmatrix}$ we need to place $1\frac{2}{3}$ g at each of A & B and $5\frac{2}{3}$ g at C.

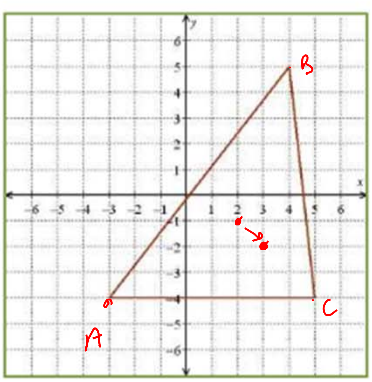


Figure 2: lamina of uniform density and thickness

m rows, n columns

Definition

The product of an $m \times n$ matrix, A, and $n \times 1$ vector, $\vec{x}$, is defined by $A\vec{x} = \sum_{i=1}^{n} [x_i \, \vec{a}_i]$ where x_i is the entry in the i^{th} row of $\vec{x}$ and $\vec{a}_i$ is the i^{th} column of A (treating the columns of A as vectors). You cannot find the product $A\vec{x}$ unless the number of columns of A is equal to the number of rows of $\vec{x}$

Example

rows x columns

Let $A = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$. Find each of the following products (where possible): $A\vec{u}$, $A\vec{v}$, $B\vec{u}$, and $B\vec{v}$.

The first step in multiplying matrices is dimensional analysis.

$A\vec{u}$ $\quad 2 \times 3 \quad 2 \times 1$ inside dimensional mismatch, the product is not possible

$A\vec{v}$ $\quad 2 \times 3 \quad 3 \times 1$ inside dimensional agreement, so there is a product and the product is 2×1

$$A\vec{v} = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix} = (-2)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (4)\begin{bmatrix} -6 \\ 5 \end{bmatrix} + 8\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(2) + (4)(-6) + (8)(-1) \\ (-2)(0) + (4)(5) + (8)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -36 \\ 36 \end{bmatrix}$$

$B\vec{u}$ $\quad 3 \times 2 \quad 2 \times 1$, so $B\vec{u}$ is 3×1

$$B\vec{u} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$= (10)\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + (2)\begin{bmatrix} -3 \\ 3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} (10)(1) + (2)(-3) \\ (10)(4) + (2)(3) \\ (10)(-2) + (2)(7) \end{bmatrix} = \begin{bmatrix} 4 \\ 46 \\ -6 \end{bmatrix}$$

$B\vec{v}$ $\quad 3 \times 2 \quad 3 \times 1$

no product

$$\begin{bmatrix} 1 & -3 \\ 4 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$$

Example

$A \vec{x} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$

Write $\begin{bmatrix} -1 & 2 & 5 \\ 8 & -2 & 0 \\ 1 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ as a vector equation.

note: Our implicit goal is to express $\begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ as a linear combination of the column vectors of A

$$x_1 \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$$

note: this in turn gives us the system of equations

$$\begin{cases} -x_1 + 2x_2 + 5x_3 = 7 \\ 8x_1 - 2x_2 = 11 \\ x_1 - 3x_2 + 6x_3 = -3 \end{cases}$$

Example

Write the system $\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 + x_4 = 9 \end{cases}$ as a matrix equation of form $a\vec{x} = \vec{b}$.

$$\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 + x_4 = 9 \end{cases} \Rightarrow x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 1 & -2 \\ 5 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

System of equations $\Longleftrightarrow$ vector equation
$\Longleftrightarrow$ matrix equation

encode $f(x) = 3 + 7x - 9x^3$ as $\vec{v} = \begin{bmatrix} 3 \\ 7 \\ 0 \\ -9 \end{bmatrix}$

Top 20 definition

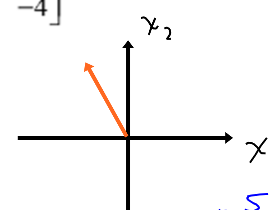
The equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ can be expressed as a linear combination of the columns of A; that is, $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is in the span of the columns of A.

Later in the term the span of the columns of A will be defined as the ***column space*** of A. It follows that the equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is in the column space of A.

Example

Describe the column spaces of the matrixes A, B, and C where $A=\begin{bmatrix}-2\\5\end{bmatrix}$, $B=\begin{bmatrix}0&0\\2&-1\\3&-4\end{bmatrix}$, and $C=\begin{bmatrix}-1&2\\3&-4\\6&7\\-1&1\end{bmatrix}$.

$$\operatorname{Col}(A) = \left\{ c_1\begin{bmatrix}-2\\5\end{bmatrix} \,\middle|\, c_1 \in \mathbb{R} \right\}$$



This set forms a line in the x_1x_2-plane through the origin with a slope of $-\frac{5}{2}$

$$\operatorname{Col}(B) = \left\{ c_1\begin{bmatrix}0\\2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\-4\end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

This column space is a plane through the origin that is perpendicular to $\begin{bmatrix}0\\2\\3\end{bmatrix} \times \begin{bmatrix}0\\-1\\-4\end{bmatrix}$

$$\operatorname{Col}(C) = \left\{ c_1\begin{bmatrix}-1\\3\\6\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\-4\\7\\1\end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

even though we cannot see nor even imagine it, this column space forms a plane.