A ***column vector*** is a matrix with only one column. Assuming that we limit the entries to real numbers, the set of all 2 x 1 (column) vectors is called $\mathbb{R}^2$ and the set of all 3 x 1 vectors is called $\mathbb{R}^3$.

Scalar multiplication is the process of multiplying a vector by a real number; the process is affected by multiplying each entry in the vector by the scalar.

Vector addition and ***vector subtraction*** are affected by adding or subtracting the corresponding entries of the two vectors; both of these operations can only be performed between vectors with the same number of rows.

Example

Simplify $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$ and illustrate the process on Figure 1.

Define $\vec{u} = \begin{bmatrix}-1\\-2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}0\\-2\end{bmatrix}$

$$-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}0\\-6\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$$

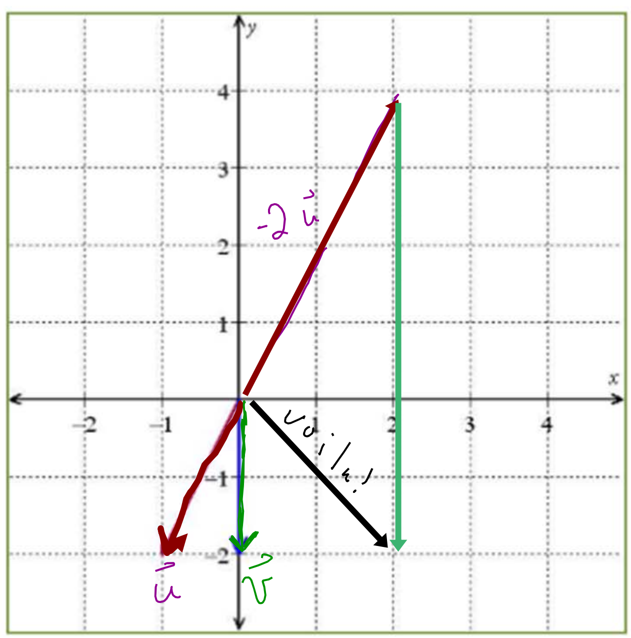


Figure 1: $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$

triple the length

rotates 180°

doubles the length

top ten vocabulary expressions

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$. We say that $\vec{y}$ is a ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ if and only if there exist scalars, $c_1, c_2, \ldots, c_p$ such that $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_p\vec{v}_p$. If such scalars exist, they are called the ***weights*** in the linear combination.

Example

Check $c_1 = \frac{7}{2}, c_2 = -1$

Let $\vec{a}_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$. Express $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

Translation: Find c_1 and c_2 so that

$$c_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\begin{bmatrix} 2c_1 \\ -4c_1 \end{bmatrix} + \begin{bmatrix} -5c_2 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\begin{bmatrix} 2c_1 - 5c_2 \\ -4c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\begin{cases} 2c_1 - 5c_2 = 12 \\ -4c_1 - c_2 = -13 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -5 & 12 \\ -4 & -1 & -13 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & -1 \end{array}\right]$$

Aha! $\vec{a}_1$ $\vec{a}_2$ $\vec{b}$

$$\therefore \vec{b} = \frac{7}{2}\vec{a}_1 + (-1)\vec{a}_2$$

[The weight on $\vec{a}_1$ is $\frac{7}{2}$ and the weight on $\vec{a}_2$ is -1].

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the ***span*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Top ten vocabulary term.

The last question could have been posed thus:
"Show that $\vec{b}$ is in the span of $\{\vec{a}_1, \vec{a}_2\}$.

Check $\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & -28/11 \\ -1 & 4 & 2 & 2 \end{array}\right] \overset{\text{RREF}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ \text{no} & & \text{inconsistency} & \end{array}\right]$

Example

Let $A = \begin{bmatrix} 3 & -1 & 5 \\ -2 & 0 & -4 \\ -1 & 4 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$. Find the value of h if $\vec{b}$ is in the span of the columns of A.

Find the value of h so that $\vec{b}$ is in the column space of A.

Translation (using x's as weights):

Make sure that

$$x_1 \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$$

has no inconsistencies.

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & h \\ -1 & 4 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ -2 & 0 & -4 & h \\ 3 & -1 & 5 & 3 \end{array}\right]$$

$$\begin{array}{l} -2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & -4+h \\ 0 & 11 & 11 & 9 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{8} R_2 \to R_2 \\ \frac{1}{11} R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -1 & -1 & -\frac{1}{2} + \frac{1}{8}h \\ 0 & 1 & 1 & \frac{9}{11} \end{array}\right]$$

$$R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -1 & -1 & -\frac{1}{2} + \frac{1}{8}h \\ 0 & 0 & 0 & \frac{7}{22} + \frac{1}{8}h \end{array}\right]$$

The system will have a solution iff

$$\frac{7}{22} + \frac{1}{8}h = 0$$

$\therefore$ The system will have a solution iff $h = -\frac{28}{11}$

Application: Center of Mass

The center of mass (or center of gravity) of an object is the average of the product of mass points in the object with their relative distance from a fixed reference point. The concept is most easily understood if on thinks about a teeter-totter. When two people of different mass sit on a teeter totter of uniform density, the center of mass is clearly going to be closer to the heavier person than the lighter person; this phenomenon is reflected in the fact that if the teeter-totter is to stay in balance, the heavier person needs to sit closer to the axis-point than the lighted person. Unfortunately, we can't define the center of mass in all cases as the balance point, because the center of mass is frequently not even on the object! This is easy to see if you think about washer (as in nuts and bolts) of uniform density; the center of mass is clearly the center point of the hole in the middle of the washer.

For most objects, it takes a double or triple integral to calculate the center of mass. In some simple situations, however, the point can be determined by a simple formula. For example, if we have a triangular lamina referenced to the xy-plane, the center of mass ($\vec{v}$) can be determined using the formula $\vec{v} = \frac{1}{3}\left[\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right]$ where $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ represent the vectors from the origin to the three vertices of the triangle; for ease of reference, we shall contextually refer to these vectors as points.

That formula is a special case of the more general formula for the center of mass of several mass points.

The center of mass ($\vec{v}$) of n mass points is: $\vec{v} = \frac{1}{m}\sum_{i=1}^{n}\left[m_i\, v_i\right]$ where m_i is the mass at point v_i and m is the sum of all the masses.

Example

Find the center of mass of the triangular lamina outlined in Figure 2 assuming that the lamina has uniform density.

$$\text{C of mass} = \frac{1}{\text{total additional mass}} \sum_{i=1}^{3} \vec{u}_i$$

$$= \frac{1}{3}\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

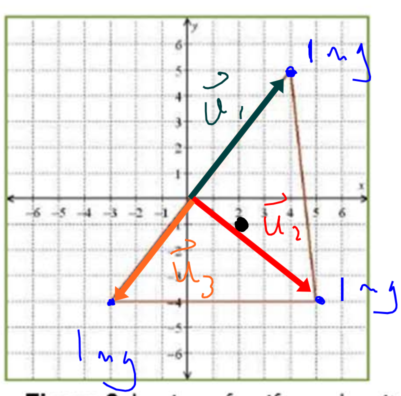


Figure 2: lamina of uniform density and thickness

Example

In the last example we found the center of mass using the formula $\vec{v} = \frac{1}{m}\sum_{i=1}^{n}[m_i v_i]$ and assuming that there was 1 g of mass at each of the vertices. Suppose that we had 9 additional grams of mass to distribute among the vertices. How should the mass be distributed so that the center of mass of the lamina shifts to the point $(3,-2)$?

Let m_i be the total mass placed a vertex v_i.

Then the center of mass will be

$$\frac{1}{12}\left[m_1\begin{bmatrix}4\\5\end{bmatrix} + m_2\begin{bmatrix}5\\-4\end{bmatrix} + m_3\begin{bmatrix}-3\\-4\end{bmatrix}\right]$$

9 additional + origal 3

and we want that to sum to $\begin{bmatrix}3\\-2\end{bmatrix}$.

$$\begin{cases}\frac{1}{12}(4m_1+5m_2-3m_3)=3\\ \frac{1}{12}(5m_1-4m_2-4m_3)=-2\end{cases}$$

$$\Rightarrow \begin{cases}4m_1+5m_2-3m_3=36\\ 5m_1-4m_2-4m_3=-24\end{cases}$$

$$\left[\begin{array}{ccc|c}4&5&-3&36\\5&-4&-4&-24\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&-32/41&24/41\\0&1&1/41&276/41\end{array}\right]$$

yeah – there seems to be more than one solution; did I overlook something?

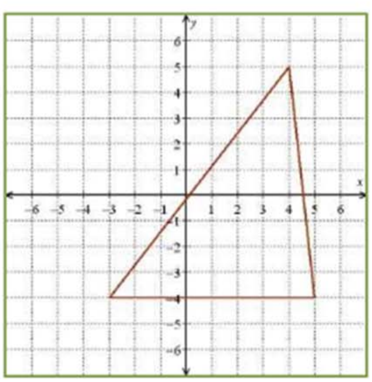


Figure 2: lamina of uniform density and thickness

D'oh! $m_1+m_2+m_3=12$ ← There's a third equation!

$$\left[\begin{array}{ccc|c}4&5&-3&36\\5&-4&-4&-24\\1&1&1&12\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&8/3\\0&1&0&20/3\\0&0&1&8/3\end{array}\right]$$

$\therefore$ we need to add $\frac{5}{3}$ gm to $\begin{bmatrix}4\\5\end{bmatrix}$, $\frac{17}{3}$ g to $\begin{bmatrix}5\\-4\end{bmatrix}$ and $5/3$ g to $\begin{bmatrix}-3\\-4\end{bmatrix}$

Definition

The product of an $m \times n$ matrix, A, and $n \times 1$ vector, $\vec{x}$, is defined by $A\vec{x} = \sum_{i=1}^{n} \left[x_i \, \vec{a}_i \right]$ where x_i is the entry in the i^{th} row of $\vec{x}$ and $\vec{a}_i$ is the i^{th} column of A (treating the columns of A as vectors). You cannot find the product $A\vec{x}$ unless the number of columns of A is equal to the number of rows of $\vec{x}$

Example

Let $A = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$. Find each of the following products (where possible): $A\vec{u}$, $A\vec{v}$, $B\vec{u}$, and $B\vec{v}$.

\# of rows $m \times n$ \# of columns

agree

$2 \times 3 \quad 3 \times 1 \qquad 2 \times 1$

$A\vec{v}$ (mental dimensional analysis: $A \quad \vec{v}$ product

The columns of A are $\vec{a}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -6 \\ 5 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

The entries in rows i of $\vec{v}$ are $x_1 = -2, x_2 = 4, x_3 = 8$.

$$A\vec{v} = \sum_{i=1}^{3} \left[x_i \vec{a}_i \right]$$

$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

$$\begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -6 \\ 5 \end{bmatrix} + 8 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -36 \\ 36 \end{bmatrix}$$

$A\vec{u}$ dimensional Analysis $2 \times 3 \quad 2 \times 1$

mission aborted/mismatched dimension

$$\begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

no scalar for the third column.

good to go

$3 \times 2 \quad 2 \times 1$

$$B\vec{u} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 46 \\ -6 \end{bmatrix}$$

$3 \times 2 \quad 3 \times 1$ $B\vec{v}$ can't be computed.

Buzz!

Example

Write $\begin{bmatrix} -1 & 2 & 5 \\ 8 & -2 & 0 \\ 1 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ as a vector equation.

$$x_1 \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$$

Example

Write the system $\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 \quad\quad + x_4 = 9 \end{cases}$ as a matrix equation of form $a\vec{x} = \vec{b}$.

$$x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

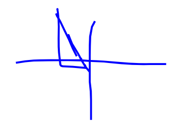
$$\begin{bmatrix} 2 & -3 & 1 & -2 \\ 5 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

$2 \times \underbrace{4 \quad 4}_{\text{good to go}} \times 1$ Result is 2×1

The equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ can be expressed as a linear combination of the columns of A; that is, $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is in the span of the columns of A. Later in the term the span of the columns of A will be defined as the ***column space*** of A. It follows that the equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is in the column space of A.

Example

Describe the column spaces of the matrixes A, B, and C where $A=\begin{bmatrix}-2\\5\end{bmatrix}$, $B=\begin{bmatrix}0&0\\2&-1\\3&-4\end{bmatrix}$, and



$C=\begin{bmatrix}-1&2\\3&-4\\6&7\\-1&1\end{bmatrix}$.

$$\text{col}(A) = \left\{ x_1 \begin{bmatrix}-2\\5\end{bmatrix} \;\middle|\; x_1 \in \mathbb{R} \right\}$$

Collectively, this set forms the line in the x_1x_2-plane through the origin with a slope of $-\frac{5}{2}$.

$$\text{col}(B) = \left\{ x_1 \begin{bmatrix}0\\2\\3\end{bmatrix} + x_2 \begin{bmatrix}0\\-1\\-4\end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}$$

Collectively, this is a plane through the origin perpendicular to $\begin{bmatrix}0\\2\\3\end{bmatrix} \times \begin{bmatrix}0\\-1\\-4\end{bmatrix}$.

$$\text{col}(C) = \left\{ x_1 \begin{bmatrix}-1\\3\\6\\-1\end{bmatrix} + x_2 \begin{bmatrix}2\\-4\\7\\1\end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}$$

Even though the vectors come from $\mathbb{R}^4$, because there are only two of them they span a plane.