A ***column vector*** is a matrix with only one column. Assuming that we limit the entries to real numbers, the set of all 2 x 1 (column) vectors is called $\mathbb{R}^2$ and the set of all 3 x 1 vectors is called $\mathbb{R}^3$.

Scalar multiplication is the process of multiplying a vector by a real number; the process is effected by multiplying each entry in the vector by the scalar.

Vector addition and ***vector subtraction*** are effected by adding or subtracting the corresponding entries of the two vectors; both of these operations can only be preformed between vectors with the same number of rows.

Example

Simplify $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$ and illustrate the process on Figure 1.

$$-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}0\\-6\end{bmatrix}$$

$$=\begin{bmatrix}2+0\\4+(-6)\end{bmatrix}$$

$$=\begin{bmatrix}2\\-2\end{bmatrix}$$

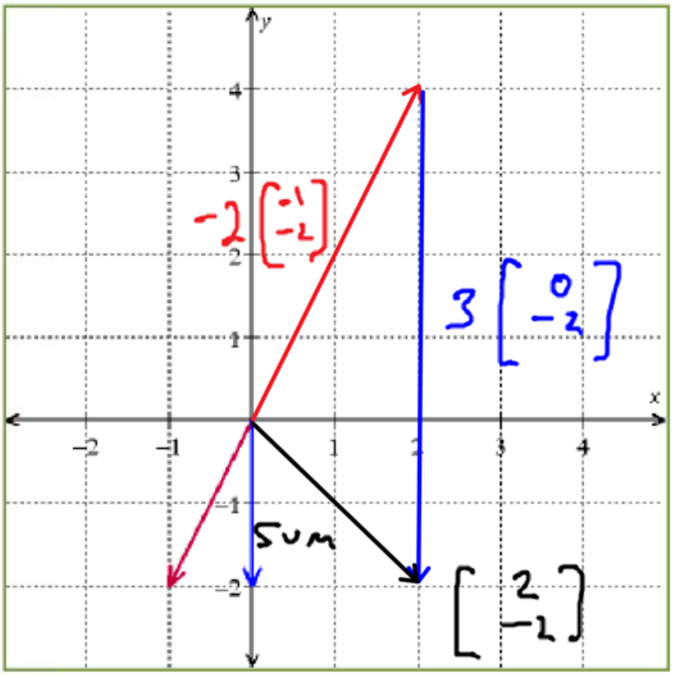


Figure 1: $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$

Importance meter: 10/10

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$. We say that $\vec{y}$ is a ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ if and only if there exist scalars, $c_1, c_2, \ldots, c_p$ such that $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_p\vec{v}_p$. If such scalars exist, they are called the ***weights*** in the linear combination.

Example

Let $\vec{a}_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$. Express $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

Translation: Find c_1 & c_2 so that

$$\vec{b} = c_1\vec{a}_1 + c_2\vec{a}_2$$

for simplicity, let's solve

$$x_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix} \quad \star$$

$$\Rightarrow \begin{bmatrix} 2x_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + (-5x_2) \\ -4x_1 + (-x_2) \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 - 5x_2 = 12 \\ -4x_1 - x_2 = -13 \end{cases} \quad \star$$

$$\left[\begin{array}{cc|c} 2 & -5 & 12 \\ -4 & -1 & -13 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & -1 \end{array}\right]$$

$$\therefore \begin{bmatrix} 12 \\ -13 \end{bmatrix} = \frac{7}{2}\begin{bmatrix} 2 \\ -4 \end{bmatrix} + (-1)\begin{bmatrix} -5 \\ -1 \end{bmatrix} \checkmark$$

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the ***span*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Example

Let $A = \begin{bmatrix} 3 & -1 & 5 \\ -2 & 0 & -4 \\ -1 & 4 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$. Find the value of h if $\vec{b}$ is in the span of the columns of A.

We want there to be at least one solution to:

$$x_1 \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & h \\ -1 & 4 & 2 & 2 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ -2 & 0 & -4 & h \\ 3 & -1 & 5 & 3 \end{array}\right]$$

$$\begin{array}{l} 3R_1 + R_3 \to R_3 \\ -2R_1 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & -4+h \\ 0 & 11 & 11 & 9 \end{array}\right]$$

$$\frac{11}{8}R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & -4+h \\ 0 & 0 & 0 & 9 + \frac{11}{8}(-4+h) \end{array}\right]$$

The system will have a solution iff

$$9 + \frac{11}{8}(-4+h) = 0$$

$$9 + \frac{11}{8}(-4+h) = 0 \Longrightarrow 72 + 11(-4+h) = 0$$

$$\Longrightarrow h = -\frac{28}{11}$$

Definition

The product of an $m \times n$ matrix, A, and $n \times 1$ vector, $\vec{x}$, is defined by $A\vec{x} = \sum_{i=1}^{n} x_i \vec{a}_i$ where x_i is the entry in the i^{th} row of $\vec{x}$ and $\vec{a}_i$ is the i^{th} column of A (treating the columns of A as vectors). You cannot find the product $A\vec{x}$ unless the number of columns of A is equal to the number of rows of $\vec{x}$.

Example

Let $A = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$. Find each of the following products (where possible): $A\vec{u}$, $A\vec{v}$, $B\vec{u}$, and $B\vec{v}$.

Dimensional analysis

$A\vec{u}$: $\underbrace{2 \times 3 \quad 2 \times 1}$ disagreement ABORT!

$A\vec{v}$: $\underbrace{2 \times 3 \quad 3 \times 1}$ agree Product exists and is 2×1

$B\vec{u}$: $3 \times 2 \quad 2 \times 1$ Product is 3×1

$B\vec{v}$: $3 \times 2 \quad 3 \times 1$ ABORT!

$$A\vec{v} = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$$

$$= -2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -6 \\ 5 \end{bmatrix} + 8\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(2) + 4(-6) + 8(-1) \\ (-2)(0) + 4(5) + 8(2) \end{bmatrix}$$

dot product of R_1 of A and column 1 of $\vec{v}$.

dot product of R2 of A and C1 of $\vec{v}$

$$= \begin{bmatrix} -36 \\ 36 \end{bmatrix}$$

$$B\vec{u} = \begin{bmatrix} (1)(10) + (-3)(2) \\ (4)(10) + (3)(2) \\ (-2)(10) + (7)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 46 \\ -6 \end{bmatrix}$$

2) Is $\begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ in the span of the column vectors of the matrix.

3) Is $\begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ in the column space of the matrix

Example

Write $\begin{bmatrix} -1 & 2 & 5 \\ 8 & -2 & 0 \\ 1 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ as a vector equation.

Three versions of the same question.

$$x_1 \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$$

This equation poses three questions:

1) Is $\begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ a linear combination of the column vectors in the matrix?

Example

Write the system $\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 \qquad + x_4 = 9 \end{cases}$ as a matrix equation of form $a\vec{x} = \vec{b}$.

$$\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 \qquad + x_4 = 9 \end{cases}$$

$$\Longrightarrow x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & -3 & 1 & -2 \\ 5 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

So important

The equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ can be expressed as a linear combination of the columns of A; that is, $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is in the span of the columns of A.
Later in the term the span of the columns of A will be defined as the ***column space*** of A. It follows that the equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is in the column space of A.

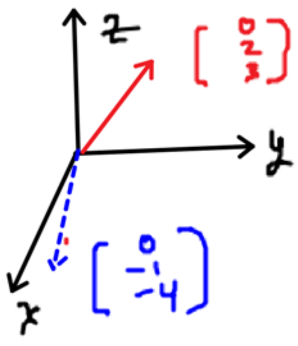
Example

Describe the column spaces of the matrixes A, B, and C where $A=\begin{bmatrix}-2\\5\end{bmatrix}$, $B=\begin{bmatrix}0&0\\2&-1\\3&-4\end{bmatrix}$, and $C=\begin{bmatrix}-1&2\\3&-4\\6&7\\-1&1\end{bmatrix}$.

The column space of A is:

$\left\{x_1\begin{bmatrix}-2\\5\end{bmatrix} \,\middle|\, x_1\in\mathbb{R}\right\}$. Geometrically this is a line in the xy-plane, through the origin with a slope of $-\frac{5}{2}$.

The column-space of B is $\left\{x_1\begin{bmatrix}0\\2\\3\end{bmatrix}+x_2\begin{bmatrix}0\\-1\\-4\end{bmatrix} \,\middle|\, x_1,x_2\in\mathbb{R}\right\}$

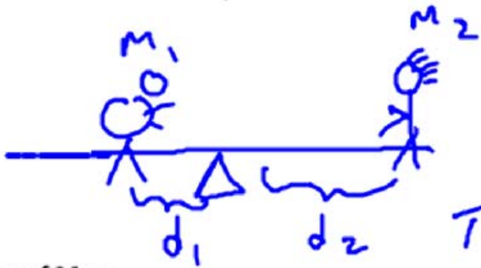


The column-space of B is the yz-plane.

The column-space of C is

$\left\{x_1\begin{bmatrix}-1\\3\\6\\-1\end{bmatrix}+x_2\begin{bmatrix}2\\-4\\7\\1\end{bmatrix} \,\middle|\, x_1,x_2\in\mathbb{R}\right\}$

We call the span of any two non-parallel vectors a plane, but I don't presume we'd recognize this as a plane.



Totter is balanced if $m_1 d_1 = m_2 d_2$

Application: Center of Mass

The center of mass (or center of gravity) of an object is the average of the product of mass points in the object with their relative distance from a fixed reference point. The concept is most easily understood if on thinks about a teeter-totter. When two people of different mass sit on a teeter totter of uniform density and thickness, the center of mass is clearly going to be closer to the heavier person than the lighter person; this phenomenon is reflected in the fact that if the teeter-totter is to stay in balance, the heavier person needs to sit closer to the tipping-point than the lighter person. Unfortunately, we can't define the center of mass in all cases as the balance point, because the center of mass is frequently not even on the object! This is easy to see if you think about a washer (as in nuts and bolts) of uniform density; the center of mass is clearly the center point of the hole in the middle of the washer.

For most objects, it takes a double or triple integral to calculate the center of mass. In some simple situations, however, the point can be determined by a simple formula. For example, if we have a triangular lamina of uniform mass-density referenced to the xy-plane, the center of mass ($\vec{v}$) can be determined using the formula $\vec{v} = \frac{1}{3}\left[\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right]$ where $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ represent the vectors from the origin to the three vertices of the triangle; for ease of reference, we shall contextually refer to these vectors as points.

That formula is a special case of the more general formula for the center of mass of several mass points.

The center of mass ($\vec{v}$) of n mass points is: $\vec{v} = \frac{1}{m}\sum_{i=1}^{n} m_i v_i$ where m_i is the mass at point v_i and m is the sum of all the masses.

Example

Find the center of mass of the triangular lamina outlined in Figure 2 assuming that the lamina has uniform density and thickness.

$$\vec{v} = \frac{1}{3}\left(\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix}-3\\-4\end{bmatrix} + \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}5\\-4\end{bmatrix}\right)$$

$$= \begin{bmatrix}2\\-1\end{bmatrix}$$

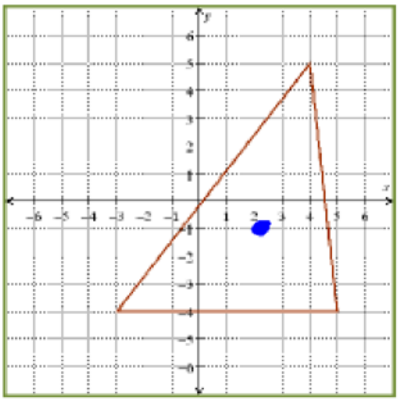


Figure 2: lamina of uniform density and thickness

Example

In the last example we found the center of mass using the formula $\vec{v} = \frac{1}{m}\sum_{i=1}^{n} m_i v_i$ and assuming that there was 1 g of mass at each of the vertices. Suppose that we had 9 addition grams of mass to distribute among the vertices. How should the mass be distributed so that the center of mass of the lamina shifts to the point $(3, -2)$?

Let w_1, w_2, w_3 represent the amount of mass (g) distributed respectively to vertices A, B + C.

① $w_1 + w_2 + w_3 = 9$

② $\frac{1}{m}\sum_{i=1}^{3} m_i v_i = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$\frac{1}{12}\left[(1+w_1)\begin{bmatrix} -3 \\ -4 \end{bmatrix} + (1+w_2)\begin{bmatrix} 4 \\ 5 \end{bmatrix} + (1+w_3)\begin{bmatrix} 5 \\ -4 \end{bmatrix}\right] = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{cases} -3w_1 + 4w_2 + 5w_3 = 30 \\ -4w_1 + 5w_2 - 4w_3 = -21 \end{cases}$$

$$\left[\begin{array}{ccc|c} -3 & 4 & 5 & 30 \\ -4 & 5 & -4 & -21 \\ 1 & 1 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 17/3 \end{array}\right]$$

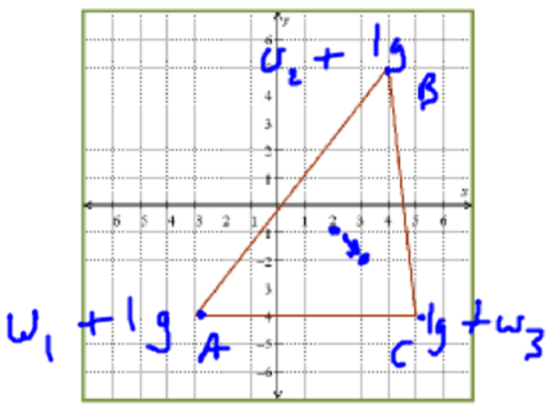


Figure 2: lamina of uniform density and thickness

∴ We need add 5/3 g to vertices A + B and 17/3 g to vertex C.

Check $\frac{8}{3}A + \frac{8}{3}B + \frac{20}{3}C = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ ✓