1. $\vec{v}_3 = 2\vec{v}_1 - 4\vec{v}_2 \;\Rightarrow\; -2\vec{v}_1 + 4\vec{v}_2 + \vec{v}_3 = \vec{0}$. This means that the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has a nontrivial solution – specifically: $c_1 = -2, c_2 = 4, c_3 = 1$. If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ were linearly independent, then the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ would be the trivial solution – that is: $c_1 = c_2 = c_3 = 0$.

2. $\vec{v}_4 = -\vec{v}_1 \;\Rightarrow\; \vec{v}_1 + \vec{v}_4 = \vec{0}$. This means that the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$ has a nontrivial solution – specifically: $c_1 = 1, c_2 = 0, c_3 = 0, c_4 = 1$. If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ were linearly independent, then the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$ would be the trivial solution – that is: $c_1 = c_2 = c_3 = c_4 = 0$.

3. If $\vec{v}_2 = \vec{0}$, then the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has a nontrivial solution – for example: $c_1 = 0, c_2 = 42.8, c_3 = 0$. If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ were linearly independent, then the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ would be the trivial solution – that is: $c_1 = c_2 = c_3 = 0$.

4. There are only three possibilities for the form of the reduced echelon equivalent of the matrix $\left[\begin{array}{ccc|c} a & c & e & 0 \\ b & d & e & 0 \end{array}\right]$. Those are:

$$\left[\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

All three of these reduced matrixes correspond to consistent systems with at least one free variable. Ergo, the equation $x_1\begin{bmatrix} a \\ b \end{bmatrix} + x_2\begin{bmatrix} c \\ d \end{bmatrix} + x_3\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has an unlimited number of solutions and, similarly, the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has nontrivial solutions. Therefore, any three vectors from $\mathbb{R}^2$ must be linearly dependent.

5. If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent, then there must be at least one non-trivial solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$; that is, there must be at least one solution where not all four values of c_i are zero. Let's assume that $c_4 \neq 0$. (This is a valid assumption because, if necessary, we could reorder the vectors so that this is true.)

   With $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$ and $c_4 \neq 0$ we get $\vec{v}_4 = -\frac{c_1}{c_4}\vec{v}_1 - \frac{c_2}{c_4}\vec{v}_2 - \frac{c_3}{c_4}\vec{v}_3$. QED

6. Suppose that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent. Suppose further that one of the vectors in the set could be written as a linear combination of the remaining vectors – as in the last example, we can assume that $\vec{v}_4$ can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. The second assumption gives us $\vec{v}_4 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ which implies that $-c_1\vec{v}_1 - c_2\vec{v}_2 - c_3\vec{v}_3 + \vec{v}_4 = \vec{0}$. But the last equation contradicts our assumption that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent. QED

7. Suppose that the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and that $\vec{v}_3$ is not in the span of $\{\vec{v}_1, \vec{v}_2\}$. If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then there is a solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ where not all three constants are zero. But c_3 must be zero or we would be able to write $\vec{v}_3 = -\frac{c_1}{c_3}\vec{v}_1 - \frac{c_2}{c_3}\vec{v}_2$ contradicting our assumption that $\vec{v}_3$ is not in the span of $\{\vec{v}_1, \vec{v}_2\}$. Since c_3 must equal zero, then if $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has a nontrivial solution, then it must be the case that $c_1\vec{v}_1 + c_2\vec{v}_2 + \overline{0} = \vec{0}$ where either c_1 and/or c_2 is not zero. But that contradicts our assumption that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. QED

8. $\vec{u} = -2\vec{x}_1 - \vec{x}_2$, $\vec{v} = 3\vec{x}_1 + \vec{x}_2$, and $\vec{w} = 3\vec{x}_1 - 2\vec{x}_2$