Definitions and a Theorem

A ***homogeneous system of equations*** is a system that can be written in the form $A\vec{x} = \vec{0}$.

Every homogeneous system of equations has at least one solution ($\vec{0}$); $\vec{0}$ is called the ***trivial solution*** to a homogeneous system of equations. Any other solution to a homogenous system of equations is called a ***nontrivial solution***.

Example

Determine whether or not each of the following is a homogenous system of equations.

a. $\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$

b. $4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c. 

Example

Describe the solution set to the homogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Example

Compare the solution set from the last example with the solution set to the nonhomogenous system

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}.$$

Example

Compare and contrast the solution sets to the homogenous system $\{3x_1 - 12x_2 - 6x_3 = 0$ and the nonhomogenous system $\{3x_1 - 12x_2 - 6x_3 = -15$.

Linear Independence vs. Linear Dependence

The set of vectors $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ is said to be ***linearly independent*** if and only if the only solution to the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}$ is $c_1 = c_2 = \cdots c_n = 0$. The set is said to be ***linearly dependent*** if there is a nontrivial solution to the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}$.

Example

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

Example

Determine the values of a that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

A bevy of facts

- A set of vectors containing more than n vectors from $\mathbb{R}^n$ must be linearly dependent.
- A set of **<u>two</u>** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector.
- A set of n vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent.
- If A is an m x n matrix, then the columns of A span $\mathbb{R}^m$ if and only if A has a pivot position in every row.

Example

Prove the second fact found in the box above.

Example

Determine whether or not the columns of A span $\mathbb{R}^3$ where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$.

Example

Consider the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. Explain why the set cannot possibly span $\mathbb{R}^3$. Afterwards, add a vector to the set so that it does span $\mathbb{R}^3$.

Example

Determine which of the following sets are linearly independent; explain!

a. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$