Definitions and a Theorem

A ***homogeneous system of equations*** is a system that can be written in the form $A\vec{x}=\vec{0}$.

Every homogeneous system of equations has at least one solution ($\vec{0}$); $\vec{0}$ is called the ***trivial solution*** to a homogeneous system of equations. Any other solution to a homogenous system of equations is called a ***nontrivial solution***.

Only homogenous system of equations have the trivial solution.

Example

Determine whether or not each of the following is a homogenous system of equations.

a. $\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$

Properly ordered, this system is

$$\begin{cases} 9x_1 + x_2 = 0 \\ 5x_1 - 2x_2 = 0 \end{cases} \quad \text{so} \quad \begin{bmatrix} 9 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

The system is homogenous.

Bonus: $\begin{bmatrix} 9 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 9 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a solution (called trivial).

b. $4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

this is equivalent to the system

$$\begin{cases} 4x_1 + 2x_1 = 0 \\ 4x_2 + 10 = 0 \\ 4x_3 + 2x_3 = 0 \end{cases} \quad \text{which begets} \quad \begin{cases} 6x_1 = 0 \\ 4x_2 = -10 \\ 6x_3 = 0 \end{cases}$$

$$\text{So} \quad \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}}_{\neq \vec{0}}$$

The system is not homogenous.

Bonus: nonhomogenous systems do not have trivial solutions.

c. 

Aw a cute kitty cat, not a homogenous system!

This equation implies a homogenous system.

Example

Describe the solution set to the homogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ -3 & 1 & -7 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equivalent system is $\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \\ 0 = 0 \end{cases}$. So, solutions can be written thus:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

The solution set is the span of $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Check the spanning vector:

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (2)(-2) + (-3)(1) + (7)(1) \\ (-3)(-2) + (1)(1) + (-7)(1) \\ (4)(-2) + (0)(1) + 8(1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Example

Compare the solution set from the last example with the solution set to the nonhomogenous system

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 9 \\ -3 & 1 & -7 & -10 \\ 4 & 0 & 8 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This simplified System is $\begin{cases} x_1 \quad + 2x_3 = 3 \\ x_2 - x_3 = -1 \\ 0 = 0 \end{cases}$

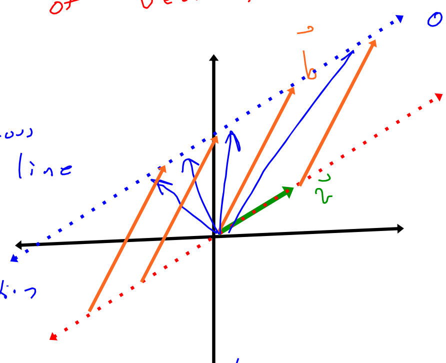
Solution can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 + 3 \\ x_3 - 1 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

note: This solution set is $\underline{\underline{\text{not}}}$ the span of a set of vectors.

In general. If the homogeneous solution is the line $\vec{x} = t\vec{v}$, the non homogenous solution is $\vec{x} = \vec{b} + t\vec{v}$ where $\vec{b}$ is any one solution.



only the heads of the solutions to the non homogenous system line on this line.

the solutions are not respectively parallel.

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Homogenous system's solution is the span of $\vec{v}$. These vectors can be drawn on that line

The nonhomogenous system's solution is a shift of the homogenous solution.

$\vec{b}$ is the shift $\vec{b} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

Example

Compare and contrast the solution sets to the homogenous system $\{3x_1 - 12x_2 - 6x_3 = 0$ and the nonhomogenous system $\{3x_1 - 12x_2 - 6x_3 = -15$.

$3x_1 - 12x_2 - 6x_3 = 0 \implies x_1 = 4x_2 + 2x_3$

So solutions have form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

The solution set is the span of $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. This is a plane through the origen (call it P)

$3x_1 - 12x_2 - 6x_3 = -15 \implies x_1 = 4x_2 + 2x_3 - 5$

So solutions are written

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 + 2x_3 - 5 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$

This is the plane P shifted so that it passes through the point $\begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$

Do all HW up through 1.6

1.6 problem 11, the 80 arrow is backwards.

Linear Independence vs. Linear Dependence

The set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be ***linearly independent*** if and only if the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ is $c_1 = c_2 = \cdots c_n = 0$. The set is said to be ***linearly dependent*** if there is a nontrivial solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$.

Example

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

Translation: show that $x_1\vec{C}_1 + x_2\vec{C}_2 + x_3\vec{C}_3 = \vec{0}$ has a nontrivial solution.

$$x_1 \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -3 & 4 & -2 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Clearly the system has solutions. QED

To nail the case even more shut,

$$\begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

Letting $x_3 = -2$, we get:

$$(4)\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + (2)\begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is showing explicitly the dependence.

Example

Determine the values of a that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

We want a unique solution $\begin{pmatrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{pmatrix}$ to:

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} a \\ a \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & a & -2 & 0 \\ 3 & a & 3 & 0 \\ -1 & -2 & a & 0 \end{array}\right] \; R_1 \leftrightarrow R_3 \; \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 3 & a & 3 & 0 \\ 2 & a & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} 3R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & -6+a & 3a+3 & 0 \\ 0 & -4+a & 2a-2 & 0 \end{array}\right]$$

$$\frac{4-a}{-6+a} R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & -6+a & 3a+3 & 0 \\ 0 & 0 & \left(\frac{4-a}{-6+a}\right)(3a+3) + (2a-2) & 0 \end{array}\right]$$

The only way the implied system will not have a unique solution is if $\left(\frac{4-a}{-6+a}\right)(3a+3) + (2a-2) = 0$

$$(-6+a)\left[\left(\frac{4-a}{-6+a}\right)(3a+3) + (2a-2)\right] = (-6+a)\,0$$

$$(4-a)(3a+3) + (2a-2)(a-6) = 0$$

$$-a^2 - 5a + 24 = 0$$

$$0 = a^2 + 5a - 24$$

$$0 = (a+8)(a-3)$$

$$a = -8 \text{ or } a = 3$$

Let's ponder 6 ...
or let's just let $a = 6$
and see what happens:

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & 0 \\ 3 & 6 & 3 & 0 \\ -1 & -2 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

unique solution!

$\therefore$ The column vectors are linearly independent iff $a \neq -8$ and $a \neq 3$.

This is linear algebra's ultimate statement.

A bevy of facts

- A set of vectors containing more than n vectors from $\mathbb{R}^n$ must be linearly dependent.
- A set of **two** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector. This only applies to sets with two vectors.
- A set of n vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent.
- If A is an m x n matrix, then the columns of A span $\mathbb{R}^m$ if and only if A has a pivot position in every row. m is the number of rows and n is the number of columns.

Example

Prove the second fact found in the box above.

$\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent $\Longleftrightarrow c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ where at least one of c_1, c_2 is not 0.

$\Longleftrightarrow$ we can legally write at least one of the following.

$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2$ or $\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1$

QED

Example

Determine whether or not the columns of A span $\mathbb{R}^3$ where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$.

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\therefore$ The column vectors of A do span $\mathbb{R}^3$.

Minimalistic explaination. A is $3 \times n$ and has 3 pivot columns. The 4th bullet point states that this implies that the column vectors of A span $\mathbb{R}^3$. $(m=3)$

Let's answer the biggest question ... why? Regardless of vector we choose from $\mathbb{R}^3$ (eg. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$) we will get

$\left[\begin{array}{ccc|c} -4 & 1 & 6 & a \\ -1 & 1 & 4 & b \\ 7 & -1 & -3 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \Leftarrow$ There's going to be a solution to $x_1\begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1\begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$

Example

Consider the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. Explain why the set cannot possibly span $\mathbb{R}^3$. Afterwards, add a vector to the set so that it does span $\mathbb{R}^3$.

Short answer: the set does not contain three vectors

fleshed out ... $\begin{bmatrix} 2 & 1 \\ 1 & -8 \\ -4 & 3 \end{bmatrix}$ This matrix can have at most two pivot positions/columns. $2 < 3$ See bullet 4.

The two vectors are obviously linearly independent (See bullet 2 – they are not multiples of one another). If we add a third linearly independent vector, the set will then span $\mathbb{R}^3$.

$$3\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -29 \\ \text{not } 0 \end{bmatrix}$$

e.g. $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ -29 \\ 37 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$

$\begin{bmatrix} 2 & 1 & 10 \\ 1 & -8 & -29 \\ -4 & 3 & * \end{bmatrix}$ This will always reduce to $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{three pivot columns}}$ unless $* = 0$

$3(-4) + 4(3) = 0$

Example

Determine which of the following sets are linearly independent; explain!

a. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$

a. Set a contains three vectors from $\mathbb{R}^2$, the set cannot possibly be linearly independent. $x_1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ cannot possibly have only one solution – more variables than equations in the attendant system.

b. This set is so $\underline{\underline{\text{not}}}$ linearly independent because it contains a zero vector. $0\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + \underset{\text{not } 0}{53{,}702}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$