Top ten

Definitions and a Theorem

A ***homogeneous system of equations*** is a system that can be written in the form $A\vec{x} = \vec{0}$.

Every homogeneous system of equations has at least one solution ($\vec{0}$); $\vec{0}$ is called the ***trivial solution*** to a homogeneous system of equations. Any other solution to a homogenous system of equations is called a ***nontrivial solution***.

Example

Determine whether or not each of the following is a homogenous system of equations.

a. $\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$

This system can be written as:

$$\begin{cases} 9x_1 + x_2 = 0 \\ 5x_1 - 2x_2 = 0 \end{cases}$$

$$\begin{bmatrix} 9 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The given system is homogeneous.

b. $4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This system can be written

$$\begin{cases} 4x_1 + 2x_1 = 0 \\ 4x_2 + 10 = 0 \\ 4x_3 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} 6x_1 = 0 \\ 4x_2 = -10 \\ 6x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}}_{\text{not } \vec{0}}$$

This system is not homogeneous

c. 

Aw, a kitty cat.

Example

Describe the solution set to the homogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ -3 & 1 & -7 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -3/2 & 7/2 & 0 \\ -3 & 1 & -7 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right]$$

$$\begin{array}{l} 3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3/2 & 7/2 & 0 \\ 0 & -7/2 & 7/2 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$$\begin{array}{l} -\frac{2}{7}R_2 \to R_2 \\ \frac{1}{6}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3/2 & \frac{7}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$-R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & -3/2 & 7/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{3}{2}R_2 + R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In the constant column

The zeros never change

For that reason, it is common to omit the zero column when manipulating homogeneous systems.

The general solution to this system is

$$\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$$

So, any vector in the solution set can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

So the solution set is the span of $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

This span is a line in $x_1x_2x_3$-space ($\mathbb{R}^3$) through the origin parallel to $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Example This is the same coefficient matrix as the last example.

Compare the solution set from the last example with the solution set to the nonhomogenous system

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}.$$

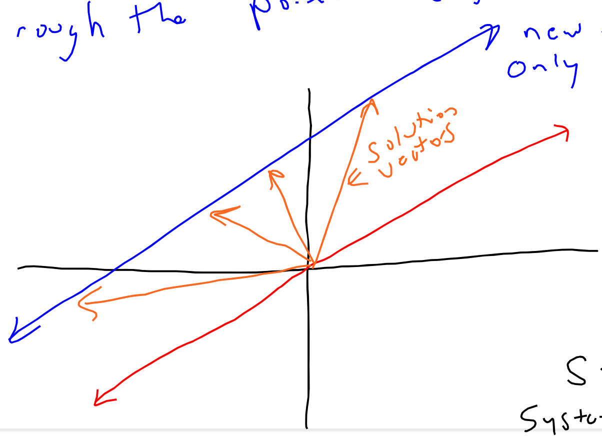
$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 9 \\ -3 & 1 & -7 & -10 \\ 4 & 0 & 8 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution can be expressed as $\begin{cases} x_1 = -2x_3 + 3 \\ x_2 = x_3 - 1 \\ x_3 \text{ is free} \end{cases}$

Vectors in this set can expressed thus:

$$\begin{bmatrix} -2x_3 + 3 \\ x_3 - 1 \\ x_3 \end{bmatrix} = 1\begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

i.e. The heads of vectors in this set lie on a line parallel to the homogenous solution shifted away from the origin so that is passes through the point $(3, -1, 0)$



new solution (nonhomogenous system) only the heads of solution vectors lie on this line.

homogenous solution: vectors that solve the homogenous solution can actually be drawn on this line

Solutions to this homogenous system are parallel; solutions to the nonhomogeneous system are not parallel.

Example

Compare and contrast the solution sets to the homogenous system $\{3x_1 - 12x_2 - 6x_3 = 0$ and the nonhomogenous system $\{3x_1 - 12x_2 - 6x_3 = -15$

Recall from 253

Initial observation: The heads of each solution set lie on planes perpendicular to $\begin{bmatrix} 3 \\ -12 \\ -6 \end{bmatrix}$. The homogeneous solution plane passes through the origin.

For the homogeneous system, solutions have form:

$[3x_1 - 12x_2 - 6x_3 = 0 \Rightarrow x_1 = 4x_2 + 2x_3]$

$$\begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

So the solution set to the homogenous system is

$$\text{Span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For the non-homogeneous system, $x_1 = 4x_2 + 2x_3 - 5$

These solutions can be written as:

$$\begin{bmatrix} 4x_2 + 2x_3 - 5 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

This solution set cannot be expressed as the span of a set of vectors.

Linear Independence vs. Linear Dependence #1

The set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be ***linearly independent*** if and only if the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ is $c_1 = c_2 = \cdots c_n = 0$. The set is said to be ***linearly dependent*** if there is a nontrivial solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$.

Example

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

Translation: Show that $x_1\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has non trivial solutions.

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -3 & 4 & -2 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution to this system is $\begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$

So, one solution is (using $x_3 = 0$) the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, but that's the trivial solution.

A non-trivial solution is (using 12) $\begin{bmatrix} -24 \\ -12 \\ 12 \end{bmatrix}$

i.e. $-24\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + (-12)\begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + 12\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

There is a non-trivial linear combination of the vectors that results in $\vec{0}$.

i.e. the vectors are linearly dependent.

Example

Determine the values of a that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

Translation: We want the only solution to $x_1\begin{bmatrix}2\\3\\-1\end{bmatrix}+x_2\begin{bmatrix}a\\a\\-2\end{bmatrix}+x_3\begin{bmatrix}-2\\3\\a\end{bmatrix}=\vec{0}$ to be $\underbrace{x_1=0,\ x_2=0,\ x_3=0}_{\text{the trivial solution}}$.

$$\left[\begin{array}{ccc|c} 2 & a & -2 & 0 \\ 3 & a & 3 & 0 \\ -1 & -2 & a & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 3 & a & 3 & 0 \\ 2 & a & -2 & 0 \end{array}\right]$$

$$\begin{array}{c} 3R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & a-6 & 3a+3 & 0 \\ 0 & a-4 & 2a-2 & 0 \end{array}\right]$$

$$\frac{1}{a-6}R_2\to R_2 \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & 1 & \frac{3a+3}{a-6} & 0 \\ 0 & a-4 & 2a-2 & 0 \end{array}\right]$$

$$-(a-4)R_2+R_3\to R_3 \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & 1 & \frac{3a+3}{a-6} & 0 \\ 0 & 0 & -(a-4)\cdot\frac{3a+3}{a-6}+(2a-2) & 0 \end{array}\right]$$

This system will have a unique solution so long as $-(a-4)\cdot\dfrac{3a+3}{a-6}+(2a-2)\neq 0$

$$-(a-4)(3a+3)+(2a-2)(a-6)\neq 0$$

$$-(3a^2-9a-12)+(2a^2-14a+12)\neq 0$$

$$-a^2-5a+24\neq 0$$

$$a^2+5a-24\neq 0$$

$$(a+8)(a-3)\neq 0$$

$\therefore$ The columns are linearly independent iff $a\neq -8$ and $a\neq 3$.

This is only true for sets of two!!

A bevy of facts

- A set of vectors containing more than n vectors from $\mathbb{R}^n$ must be linearly dependent. (1)
- A set of **two** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector. (2)
- A set of n vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent. (3)
- If A is an m x n matrix, then the columns of A span $\mathbb{R}^m$ if and only if A has a pivot position in every row. (4)

Example

Prove the second fact found in the box above.

$\vec{v}_1$ and $\vec{v}_2$ are linearly dependent $\iff c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ has a non-trivial solution where not both $c_1 = 0$ and $c_2 = 0$

$\iff c_1\vec{v}_1 = -c_2\vec{v}_2$

$\iff \vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2$ and/or $\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1$

QED

Example

Determine whether or not the columns of A span $\mathbb{R}^3$ where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$.

From (3), three vectors from $\mathbb{R}^3$ span $\mathbb{R}^3$ iff the vectors are linearly independent.

$$\left[\begin{array}{ccc|c} -4 & 1 & 6 & 0 \\ -1 & 1 & 4 & 0 \\ 7 & -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

imagine

This tells us that the only solution to $A\vec{x} = \vec{0}$ is the trivial solution

$\therefore$ The columns are linearly independent. $\therefore$ The columns span $\mathbb{R}^3$

Example

Consider the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. Explain why the set cannot possibly span $\mathbb{R}^3$. Afterwards, add a vector to the set so that it does span $\mathbb{R}^3$.

These vectors cannot span $\mathbb{R}^3$ because it takes at least three vectors to span $\mathbb{R}^3$. Those vectors are linearly independent (two non-multiple vectors are always linearly indepent). If we add a third linearly independent vector the larger set will span $\mathbb{R}^3$.

$\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \longrightarrow$ new vector $\begin{bmatrix} 2+1 \\ 1+(-8) \\ \text{not } -4+3 \end{bmatrix}$

(In 3-d $\vec{u} \times \vec{v}$ would work as well)

Example

Determine which of the following sets are linearly independent; explain!

a. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$

Set a is not linearly independent. The most vectors a subset of $\mathbb{R}^2$ can contain and maintain independence is two.

Set b is so not linearly independent.

NO set containing $\vec{0}$ is linearly independent.

eg. $0 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + \frac{11}{563} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

non-trivial solution.