Important!
Add 1.7 Hw to this week's Hw list!

MTH 261 – Mr. Simonds' class

**Definitions and a Theorem**

A ***homogeneous system of equations*** is a system that can be written in the form $A\vec{x}=\vec{0}$ .

Every homogeneous system of equations has at least one solution ($\vec{0}$ ); $\vec{0}$ is called the ***trivial solution*** to a homogeneous system of equations. Any other solution to a homogenous system of equations is called a ***nontrivial solution***.

**Example**

Determine whether or not each of the following is a homogenous system of equations.

a. $\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$

$\begin{cases} 9x_1 + x_2 = 0 \\ 5x_1 - 2x_2 = 0 \end{cases}$

$\begin{bmatrix} 9 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ all zero constant

This system is homogenous

b. $4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ looks homogenous, but it's not

Trick question

$\begin{cases} 4x_1 + 2x_1 = 0 \\ 4x_2 + 10 = 0 \\ 4x_3 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} 6x_1 = 0 \\ 4x_2 = 70 \\ 6x_3 = 0 \end{cases}$

$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}$

not $\vec{0}$

c. 

Ah! A kitty cat :)

HOMOGENOUS

**Example**

Describe the solution set to the homogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Reduced system

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ -3 & 1 & -7 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$

Solution set: $\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 = \text{free} \end{cases}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Solution set: $\left\{ x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$

The solution set is the span of $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$

All vectors in the solution set "lie" on the line through the origin parallel to $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Same coefficient matrix

**Example**

Compare the solution set from the last example with the solution set to the nonhomogenous system

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 9 \\ -3 & 1 & -7 & -10 \\ 4 & 0 & 8 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

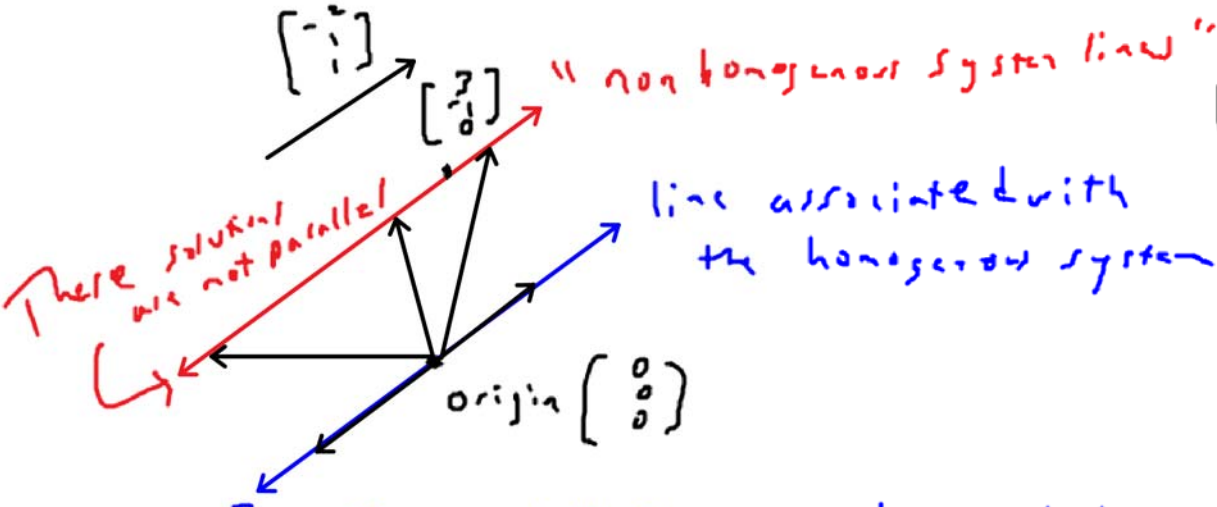
reduced system

$$\begin{cases} x_1 + 2x_3 = 3 \\ x_2 - x_3 = -1 \end{cases}$$

Solution set: $\begin{cases} x_1 = -2x_3 + 3 \\ x_2 = x_3 - 1 \\ x_3 = \text{free} \end{cases}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 + 3 \\ x_3 - 1 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

The solution set is the "line that contained the homogenous system solution set" shifted so that it passes through the "point" $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$




The solutions are all parallel

Two solutions to the nonhomogenous system are $x_3 = 0$ $x_3 = 1$

$\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

not parallel

**Example**

Compare and contrast the solution sets to the homogenous system $\{3x_1 - 12x_2 - 6x_3 = 0$ and the nonhomogenous system $\{3x_1 - 12x_2 - 6x_3 = -15$.

$$3x_1 - 12x_2 - 6x_3 = 0 \Rightarrow x_1 = 4x_2 + 2x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

solution set: $\left\{ s\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$

The solution set is "a plane through the origin parallel to $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$."

$$3x_1 - 12x_2 - 6x_3 = -15 \Rightarrow x_1 = 4x_2 + 2x_3 - 5$$

Solution set: $\left\{ s\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$

The <u>heads</u> of the solution vectors to the nonhomogenous system all lie on this shifted plane (assuming the tails are drawn at the origin).

How important is this concept? 10, baby, 10!

**Linear Independence vs. Linear Dependence**

The set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be ***linearly independent*** if and only if the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ is $c_1 = c_2 = \cdots c_n = 0$. The set is said to be ***linearly dependent*** if there is a nontrivial solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$.

**Example**

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

Translation: There are nontrivial solutions to:

$$x_1 \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -3 & 4 & -2 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution set

$$\begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 = \text{free} \end{cases}$$

Solution set: $\left\{ t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

one specific solution is $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

$$\therefore \; -2\underbrace{\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}}_{\vec{c}_1} + (-1)\underbrace{\begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}}_{\vec{c}_2} + (1)\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\vec{c}_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{QED}$$

note: One repercussion of this is that

$$\vec{c}_3 = 2\vec{c}_1 + \vec{c}_2$$

The vector $\vec{c}_3$ is dependent upon the values of $\vec{c}_1$ + $\vec{c}_2$



likewise, $\vec{c}_1$ is linearly dependent upon $\vec{c}_2$ & $\vec{c}_3$

$$\vec{c}_1 = \frac{1}{2}\vec{c}_2 - \frac{1}{2}\vec{c}_3$$

If $a=6$, our last row operation is invalid. Instead, swap $R_2$ & $R_3$ and see what happens.

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**Example**

Determine the values of $a$ that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

Translation: we want the only solution to $x_1\vec{C}_1 + x_2\vec{C}_2 + x_3\vec{C}_3 = \vec{0}$ to be $x_1 = x_2 = x_3 = 0$.

$$\left[\begin{array}{ccc|c} 2 & a & -2 & 0 \\ 3 & a & 3 & 0 \\ -1 & -2 & a & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 3 & a & 3 & 0 \\ 2 & a & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} 3R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & a-6 & 3a+3 & 0 \\ 0 & a-4 & 2a-2 & 0 \end{array}\right]$$

$$-\frac{a-4}{a-6}R_2+R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & a-6 & 3a+3 & 0 \\ 0 & 0 & 2a-2-\frac{a-4}{a-6}(3a+3) & 0 \end{array}\right]$$

If $2a-2-\frac{a-4}{a-6}(3a+3) \neq 0$, then $x_3 = 0$ which, backing up, makes $x_2 = 0$ & $x_1 = 0$.

The system is linearly dependent iff

$$(a-6)\left(2a-2-\frac{a-4}{a-6}(3a+3)\right) = (0)(a-6)$$

$$(a-6)(2a-2)-(a-4)(3a+3) = 0$$

$$-a^2-5a+24 = 0$$

$$a^2+5a-24 = 0$$

$$(a+8)(a-3) = 0$$



$\therefore$ The columns of $A$ are linearly independent so long as $a \neq -8$ and $a \neq 3$. See above

Every one of these things is HUGELY important

**A bevy of facts**

- A set of vectors containing more than $n$ vectors from $\mathbb{R}^n$ must be linearly dependent.
- A set of **two** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector.
- A set of $n$ vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent.
- If $A$ is an $m$ x $n$ matrix, then the columns of $A$ span $\mathbb{R}^m$ if and only if $A$ has a pivot position in every row. Caveat – this one isn't "fundamental!"

A set like this is called a basis of $\mathbb{R}^n$

**Example**
Prove the second fact found in the box above.

Two vectors are linearly dependent $\iff x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$ has non-trivial solutions

$\iff x_1 \neq 0$ and/or $x_2 \neq 0$

$\iff \vec{v}_1 = -\frac{x_2}{x_1}\vec{v}_2$ and/or $\vec{v}_2 = -\frac{x_1}{x_2}\vec{v}_1$

QED

**Example**

Determine whether or not the columns of $A$ span $\mathbb{R}^3$ where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$.

$$\begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns come from $\mathbb{R}^3$. In RREF, there are three rows with pivot positions.

Ergo, the columns span $\mathbb{R}^3$. (4th Bullet)

Because there are three vectors that span $\mathbb{R}^3$, they form a basis for $\mathbb{R}^3$.

$$x_1\begin{bmatrix}2\\1\end{bmatrix} + x_2\begin{bmatrix}1\\-8\end{bmatrix} = \begin{bmatrix}3\\-7\end{bmatrix}$$



$$\left[\begin{array}{cc|c}2 & 1 & 3\\1 & -8 & -7\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 1\end{array}\right]$$

**Example**

Consider the set $\left\{\begin{bmatrix}2\\1\\-4\end{bmatrix},\begin{bmatrix}1\\-8\\3\end{bmatrix}\right\}$. Explain why the set cannot possibly span $\mathbb{R}^3$. Afterwards, add a vector to the set so that it does span $\mathbb{R}^3$.

Think about this one!

The short answer is that the dimension of $\mathbb{R}^3$ is three meaning it takes at least three vectors to span $\mathbb{R}^3$. Geometrically, two vectors can span at most a plane. If we introduce a vector that is not a linear combination of the two given vectors, the three vectors will span $\mathbb{R}^3$. eg. let's add $\vec{v}_1 + \vec{v}_2$ with a mistake in the third row

$$\begin{bmatrix}2+1\\1+(-8)\\4+3\end{bmatrix} = \begin{bmatrix}3\\-7\\7\end{bmatrix}$$

mistake

Based upon the first two rows, the only way this vector is a linear combination of $\vec{v}_1 + \vec{v}_2$ is if it's $\vec{v}_1 + \vec{v}_2$. (see above)

**Example**

Determine which of the following sets are linearly independent; explain!

a. $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}-2\\3\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}\right\}$

b. $\left\{\begin{bmatrix}1\\2\\-5\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}4\\4\\21\end{bmatrix}\right\}$

a. The set cannot possibly be linearly indepent. A linearly indepent set in $\mathbb{R}^2$ cannot exceed two vectors.

b. No set containing $\vec{0}$ is linearly independent.

$$0\vec{v}_1 + 12.7\,\vec{0} + 0\,\vec{v}_3 = \vec{0}$$