Definitions and a Theorem

A ***homogeneous system of equations*** is a system that can be written in the form $A\vec{x} = \vec{0}$.

Every homogeneous system of equations has at least one solution ($\vec{0}$); $\vec{0}$ is called the ***trivial solution*** to a homogeneous system of equations. Any other solution to a homogenous system of equations is called a ***nontrivial solution***.

Example

Determine whether or not each of the following is a homogenous system of equations.

a. $\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$

$$\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases} \Rightarrow \begin{cases} 9x_1 + x_2 = 0 \\ 5x_1 - 2x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 9 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system is homogenous.

b. $4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 4x_1 + 2x_1 = 0 \\ 4x_2 + 10 = 0 \\ 4x_3 + 2x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 6x_1 = 0 \\ 4x_2 = -10 \\ 6x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}$$

This system is not homogenous

c. 

This is not a system, it is a kitty cat.

Awww ....

Example

Describe the solution set to the homogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The given system is equivalent to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ so, } \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$$

So solution vectors have form:

$$\begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The solution set is the span of the vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. This set is a line, through the origin, parallel to $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Example

Compare the solution set from the last example with the solution set to the nonhomogenous system

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 9 \\ -3 & 1 & -7 & -10 \\ 4 & 0 & 8 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The given system is equivalent to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

So ..... the solution is $\begin{cases} x_1 = 3 - 2x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ is free} \end{cases}$

Solution vectors can be written thus:

$$\begin{bmatrix} 3 - 2x_3 \\ -1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

This solution set is a line parallel to $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ shifted from the origin to the point $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$.

$\left[\begin{array}{ccc|c} 3 & -12 & -6 & 0 \end{array}\right]$

Example

Compare and contrast the solution sets to the homogenous system $\{3x_1 - 12x_2 - 6x_3 = 0$ and the nonhomogenous system $\{3x_1 - 12x_2 - 6x_3 = -15$.

Homogenous system

$3x_1 - 12x_2 - 6x_3 = 0 \implies x_1 = 4x_2 + 2x_3$

Solution vectors have form:

$$\begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is the span of $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Geometrically, this is a plane, through the origin, parallel to both $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

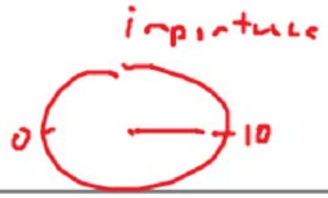
nonhomogenous system

$3x_1 - 12x_2 - 6x_3 = -15 \implies x_1 = -5 + 4x_2 + 2x_3$

solution vectors have form:

$$\begin{bmatrix} -5 + 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

This is a plane parallel to the homogenous system plane shifted from the origin to $\begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$



Linear Independence vs. Linear Dependence

The set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be ***linearly independent*** if and only if the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ is $c_1 = c_2 = \cdots c_n = 0$. The set is said to be ***linearly dependent*** if there is a nontrivial solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$.

Example

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

Translation: Show that

$$x_1 \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solutions.

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 0 \\ -3x_1 + 4x_2 - 2x_3 = 0 \\ 3x_1 - 5x_2 + x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So.... $x_1 = -2x_3$, $x_2 = -x_3$ so... a nontrivial solution is $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$\therefore\ 2\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vectors are linearly dependent!

$a=3$ $\left[\begin{array}{ccc|c} 2 & 3 & -2 & 0 \\ 3 & 3 & 3 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{cases} x_1 = -5x_3 \\ x_2 = 4x_3 \\ x_3 \text{ is free} \end{cases}$ non-trivial solutions! This makes the columns dependent!

Example

Determine the values of a that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

$$\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -1 & -2 & a \\ 3 & a & 3 \\ 2 & a & -2 \end{bmatrix}$$

$$\begin{array}{l} 3R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} -1 & -2 & a \\ 0 & a-6 & 3a+3 \\ 0 & a-4 & 2a-2 \end{bmatrix}$$

$$\frac{1}{a-6}R_2 \to R_2 \begin{bmatrix} -1 & -2 & a \\ 0 & 1 & \frac{3a+3}{a-6} \\ 0 & a-4 & 2a-2 \end{bmatrix}$$

$$-(a-4)R_2 + R_3 \to R_3 \begin{bmatrix} -1 & -2 & a \\ 0 & 1 & \frac{3a+3}{a-6} \\ 0 & 0 & 2a-2-(a-4)\left(\frac{3a+3}{a-6}\right) \end{bmatrix}$$

The last equation in our equivalent system is $\left[2a-2-(a-4)\left(\frac{3a+3}{a-6}\right)\right]x_3 = 0$

The system will have nontrivial solutions if $2a-2-(a-4)\left(\frac{3a+3}{a-6}\right) = 0$

$$2a-2-(a-4)\left(\frac{3a+3}{a-6}\right) = 0 \Rightarrow (2a-2)(a-6)-(a-4)(3a+3) = 0$$
$$\Rightarrow a = 3 \text{ or } a = -8$$

$\therefore$ The columns will be linearly independent so long as the value of a is not 3 nor -8.

A bevy of facts

- A set of vectors containing more than n vectors from $\mathbb{R}^n$ must be linearly dependent.
- A set of **two** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector.
- A set of n vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent.
- If A is an m x n matrix, then the columns of A span $\mathbb{R}^m$ if and only if A has a pivot position in every row.

Example
Prove the second fact found in the box above.

$\{\vec{v}_1, \vec{v}_2\}$ is dependent $\iff c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ has non-trivial solutions

$\iff v_1 = -\frac{c_2}{c_1}\vec{v}_2$ and/or $\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1$

QED

Example

Determine whether or not the columns of A span $\mathbb{R}^3$ where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$.

Three vectors span $\mathbb{R}^3$ iff they are linearly independent. Is the only solution to

$$x_1 \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0,\ x_2 = 0,\ x_3 = 0$?

$$\left[\begin{array}{ccc|c} -4 & 1 & 6 & 0 \\ -1 & 1 & 4 & 0 \\ 7 & -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The only solution to the dependence equation is $x_1 = 0, x_2 = 0, x_3 = 0$, so the columns do span $\mathbb{R}^3$.

Example

Consider the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. Explain why the set cannot possibly span $\mathbb{R}^3$. Afterwards, add a vector to the set so that it does span $\mathbb{R}^3$.

Example

Determine which of the following sets are linearly independent; explain!

a. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$

Set "a" <u>must</u> be dependent; three vectors from $\mathbb{R}^2$ cannot possibly be linearly independent

Set "b" <u>must</u> be dependent cuz of $\vec{0}$

e.g. $0\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + \sqrt{7}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$