Theorem

The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.

Example

Let $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$. Discuss why $U^{\mathrm{T}}U$ must equal I_2; i.e., discuss what two phenomena (other than dimensional analysis) create that outcome.

Let $\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find an orthonormal basis for $\text{span}\{\vec{x}_1, \vec{x}_2\}$. Next, express $\vec{x}_1$ and $\vec{x}_2$ in terms of that new basis.