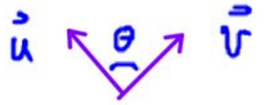
$\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$



Theorem

The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.

Example

Let $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$. Discuss why $U^{\mathrm{T}}U$ must equal I_2; i.e., discuss what two phenomena (other than dimensional analysis) create that outcome.

Let's call the columns $\vec{c}_1$ & $\vec{c}_2$ respectively.

normal ✓ $|\vec{c}_1| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$ $\qquad |\vec{c}_2| = \sqrt{\frac{1}{14}+\frac{4}{14}+\frac{9}{14}} = 1$

ortho ✓ $\vec{c}_1 \cdot \vec{c}_2 = \frac{1}{\sqrt{42}} + \frac{2}{\sqrt{42}} - \frac{3}{\sqrt{42}} = 0$

$$U^{\mathrm{T}}U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$u_{12} = u_{21} = \vec{c}_1 \cdot \vec{c}_2 = 0$

$u_{11} = \vec{c}_1 \cdot \vec{c}_1 = |\vec{c}_1||\vec{c}_1|\cos(0) = (1)(1)(1) = 1$

similarly $u_{22} = 1$

Check: $\vec{u}_1 + \frac{25}{\sqrt{125}}\vec{u}_2$

$$= \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix} + \frac{25}{\sqrt{125}}\begin{bmatrix} 6/\sqrt{125} \\ 5/\sqrt{125} \\ 8/\sqrt{125} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \vec{x}_2 \checkmark$$

Let $\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find an orthonormal basis for $\operatorname{span}\{\vec{x}_1, \vec{x}_2\}$. Next, express $\vec{x}_1$ and $\vec{x}_2$ in terms of that new basis.

Let's find any orthogonal basis for span $\{\vec{x}_1, \vec{x}_2\}$ and then normalize that. Let's Gram-Schmidt

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{25}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6/5 \\ 1 \\ 8/5 \end{bmatrix}$$

$\therefore$ An orthogonal basis for span $\{\vec{x}_1, \vec{x}_2\}$ is $\left\{ \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix} \right\}$ and an orthonormal basis is $\left\{ \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}, \begin{bmatrix} 6/\sqrt{125} \\ 5/\sqrt{125} \\ 8/\sqrt{125} \end{bmatrix} \right\}$. Call these basis vectors $\vec{u}_1$ & $\vec{u}_2$, respectively.

Obviously $\vec{x}_1 = 5\vec{u}_1 + 0\vec{u}_2$. Because we have an orthogonal basis,

$$\vec{x}_2 = \frac{\vec{x}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{x}_2 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2$$

$$= \frac{1}{1}\vec{u}_1 + \frac{25/\sqrt{125}}{1}\vec{u}_2$$