

**Projection vectors in  $\mathbb{R}^n$** 

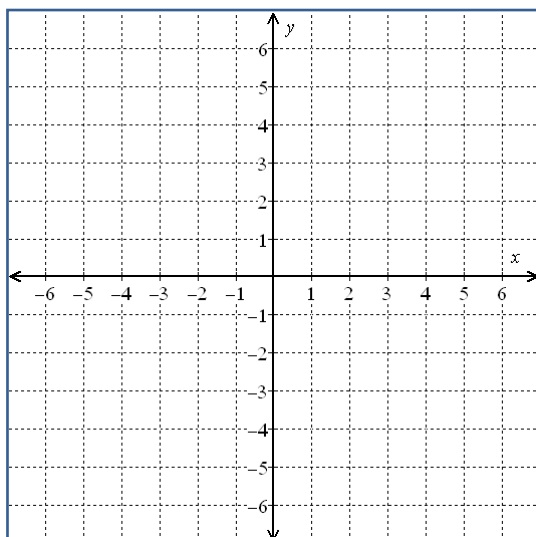
The **orthogonal projection of  $\vec{y}$  onto  $\vec{u}$**  is the vector  $\text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$ .

The **component of  $\vec{y}$  orthogonal to  $\vec{u}$**  is  $\vec{y} - \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$ .

The quantity  $\text{comp}_{\vec{u}}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$  is called the **scalar component of  $\vec{y}$  onto  $\vec{u}$** .

**Example**

Find and illustrate the projection vector, component vector, and scalar component when  $\vec{y} = [2, -4]^T$  and  $\vec{u} = [5, 1]^T$ .



**A definition and a theorem about orthogonal sets of vectors in  $\mathbb{R}^n$** 

A set of non-zero vectors,  $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ , is called an **orthogonal set** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if  $\vec{u}_i \cdot \vec{u}_j = 0 \forall i \neq j$ .

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of  $n$  vectors from  $\mathbb{R}^n$  forms a basis for  $\mathbb{R}^n$ .

**Change of coordinates in  $\mathbb{R}^n$** 

Let  $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$  be an orthogonal set from  $\mathbb{R}^n$ . Then every vector  $\vec{y}$  in  $\mathbb{R}^n$  can be expressed as  $\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_n \vec{u}_n$  where  $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$ .

That is,  $[\vec{y}]_\beta = [c_1, c_2, \dots, c_n]^T$  where  $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$ .

**Example**

Show that the set  $\beta = \left\{ [1, 1, 1]^T, [-1, -1, 2]^T, [3, -3, 0]^T \right\}$  is orthogonal and, therefore, forms a basis for  $\mathbb{R}^3$ . Then find  $[\vec{y}]_\beta$  where  $\vec{y} = [5, -2, 0]^T$ .

**The Gram-Schmidt process for finding an orthogonal basis**

Let  $\beta = \{ \vec{x}_1, \vec{x}_2, \dots, \vec{x}_p \}$  be a linearly independent set of vectors.

Then the set  $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \}$  is an orthogonal basis for the span of  $\beta$  where:

$$\begin{aligned}\vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2) \\ \vec{v}_3 &= \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \text{proj}_{\vec{v}_1}(\vec{x}_p) - \text{proj}_{\vec{v}_2}(\vec{x}_p) - \dots - \text{proj}_{\vec{v}_{p-1}}(\vec{x}_p)\end{aligned}$$

Furthermore, the set  $\gamma = \left\{ \frac{\vec{v}_1}{|\vec{v}_1|}, \frac{\vec{v}_2}{|\vec{v}_2|}, \dots, \frac{\vec{v}_p}{|\vec{v}_p|} \right\}$  is an **orthonormal basis** for the span of  $\beta$ .

**Example**

Let's find an orthonormal basis for the column space of  $A = \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix}$  and find the

coordinates of  $\vec{y} = [5, -10, -6, 7]^T$  in terms of that basis.

$$A = \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix}$$

**A definition and a trio of theorems**

If  $W$  is a subspace of  $\mathbb{R}^n$ , then the **orthogonal compliment** of  $W$  is the set of vectors,  $W^\perp$ , that are orthogonal to every vector in  $W$ .

- $W^\perp$  is a subspace of  $\mathbb{R}^n$ .
- $(\text{row}(A))^\perp = \text{nul}(A)$ .
- $(\text{col}(A))^\perp = \text{nul}(A^T)$ .

**Example**

Prove that  $W^\perp$  is a subspace of  $\mathbb{R}^n$ .

**Example**

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Illustrate that  $(\text{row}(A))^\perp = \text{nul}(A)$ .

**Example**

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Illustrate that  $(\text{col}(A))^\perp = \text{nul}(A^T)$ .

**The Orthogonal Decomposition Theorem**

Let  $W$  be a subspace of  $\mathbb{R}^n$  and  $\vec{y} \in \mathbb{R}^n$ . Then there exists unique vectors  $\vec{w}$  and  $\vec{z}$ ,  $\vec{w} \in W$ ,  $\vec{z} \in W^\perp$ , such that  $\vec{y} = \vec{w} + \vec{z}$ .

In fact, if the set  $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$  is an orthogonal basis for  $W$ , then

$$\vec{w} = \text{proj}_{\vec{u}_1}(\vec{y}) + \text{proj}_{\vec{u}_2}(\vec{y}) + \dots + \text{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \vec{z} = \vec{y} - \vec{w}$$

**Example**

Let  $\vec{u}_1 = [1 \ -1 \ -1 \ 1]^T$ ,  $\vec{u}_2 = [2 \ 1 \ 1 \ 0]^T$ , and  $\vec{u}_3 = [-1 \ 1 \ 1 \ 3]^T$ . Let  $W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  and  $\vec{y} = [2 \ 1 \ 3 \ 4]^T$ . Express  $\vec{y}$  as the sum of two vectors, one from  $W$  and the other from  $W^\perp$ .

**Example**

A basis from  $\mathbb{R}^3$  for the  $x_1 x_2$  – plane is  $\{\vec{u}_1, \vec{u}_2\}$  where  $\vec{u}_1 = [3 \ -4 \ 0]^T$  and  $\vec{u}_2 = [4 \ 3 \ 0]^T$ . Use this basis to determine the point on the  $x_1 x_2$  – plane that is closest to the point  $(2, 7, 5)$ .



**Theorem**

The  $m \times n$  matrix  $U$  has orthonormal columns if and only if  $U^T U = I_n$ .

**Example**

Let  $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$ . Discuss why  $U^T U$  must equal  $I_2$ ; i.e., discuss what two phenomena

(other than dimensional analysis) create that outcome.

Let  $\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$  and  $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ . Find an orthonormal basis for  $\text{span}\{\vec{x}_1, \vec{x}_2\}$ . Next, express  $\vec{x}_1$  and  $\vec{x}_2$  in terms of that new basis.