Projection vectors in $\mathbb{R}^n$

parallel to $\vec{u}$ ← The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

perpendicular to $\vec{u}$ ← The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y} - \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

+ add to $\vec{y}$

The quantity $\text{comp}_{\vec{u}}(\vec{y}) = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

Example

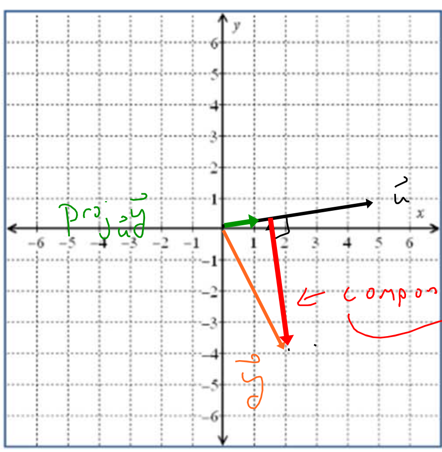
Find and illustrate the projection vector, component vector, and scalar component when $\vec{y} = [2,-4]^T$ and $\vec{u} = [5,1]^T$.

$$\left.\begin{aligned} \text{proj}_{\vec{u}}\vec{y} &= \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\ &= \frac{6}{26}\begin{bmatrix}5\\1\end{bmatrix} \\ &= \frac{3}{13}\begin{bmatrix}5\\1\end{bmatrix} \end{aligned}\right\}$$ orthogonal projection of $\vec{y}$ onto $\vec{u}$

↑ scalar component of $\vec{y}$ onto $\vec{u}$ ($\text{comp}_{\vec{u}}\vec{y} = \frac{3}{13}$)

Component of $\vec{y}$ orthogonal to $\vec{u}$ is $\underbrace{\vec{y} - \text{proj}_{\vec{u}}\vec{y}}_{\text{This has no symbol (in this book)}} = \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}15/13\\3/13\end{bmatrix}$

$= \begin{bmatrix}11/13\\-55/13\end{bmatrix}$



$\text{comp}_{\vec{u}}\vec{y}$ is ① The relative length of $\text{proj}_{\vec{u}}\vec{y}$ to $\vec{u}$ (in this case $\vec{y}$ is 3/13 in length compared to $\vec{u}$)
② The relative direction of $\text{proj}_{\vec{u}}\vec{y}$ to $\vec{u}$
+ = same
− = opposite

$\text{proj}_{\vec{u}}\vec{y} + \text{orth}_{\vec{u}}\vec{y} = \vec{y}$

A definition and a theorem about orthogonal sets of vectors in $\mathbb{R}^n$

A set of non-zero vectors, $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i \cdot \vec{u}_j = 0 \ \forall\, i \neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Change of coordinates in $\mathbb{R}^n$

Let $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ where $c_i = \mathrm{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_\beta = [c_1, c_2, \ldots, c_n]^{\mathrm{T}}$ where $c_i = \mathrm{comp}_{\vec{u}_i}(\vec{y})$.

Example

(a) Show that the set $\beta = \{\underset{\vec{u}_1}{[1,1,1]^{\mathrm{T}}}, \underset{\vec{u}_2}{[-1,-1,2]^{\mathrm{T}}}, \underset{\vec{u}_3}{[3,-3,0]^{\mathrm{T}}}\}$ is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. (b) Then find $[\vec{y}]_\beta$ where $\vec{y} = [5,-2,0]^{\mathrm{T}}$.

(a) $\vec{u}_1 \cdot \vec{u}_2 = -1 - 1 + 2 = 0$ $\quad$ $\vec{u}_1 \cdot \vec{u}_3 = 3 - 3 + 0 = 0$ $\quad$ $\vec{u}_2 \cdot \vec{u}_3 = -3 + 3 + 0 = 0$ ✓

(b) $c_1 = \mathrm{comp}_{\vec{u}_1}\vec{y} = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \dfrac{3}{3} = 1$

$c_2 = \mathrm{comp}_{\vec{u}_2}\vec{y} = \dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \dfrac{-3}{6} = -\dfrac{1}{2}$

$c_3 = \mathrm{comp}_{\vec{u}_3}\vec{y} = \dfrac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \dfrac{21}{18} = 7/6$

$$(1)\begin{bmatrix}1\\1\\1\end{bmatrix} + (-1/2)\begin{bmatrix}-1\\-1\\2\end{bmatrix} + (7/6)\begin{bmatrix}3\\-3\\0\end{bmatrix} = \begin{bmatrix}30/6\\-12/6\\0/6\end{bmatrix} = \begin{bmatrix}5\\-2\\0\end{bmatrix} = \vec{y}$$ ✓

The Gramm-Schmidt process **for finding an orthogonal basis**

Let $\beta = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \}$ be a linearly independent set of vectors.

Then the set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \}$ is an orthogonal basis for the span of β where:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2) \\ \vec{v}_3 &= \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \text{proj}_{\vec{v}_1}(\vec{x}_p) - \text{proj}_{\vec{v}_2}(\vec{x}_p) - \cdots - \text{proj}_{\vec{v}_{p-1}}(\vec{x}_p) \end{aligned}$$

Furthermore, the set $\gamma = \left\{ \dfrac{\vec{v}_1}{|\vec{v}_1|}, \dfrac{\vec{v}_2}{|\vec{v}_2|}, \ldots, \dfrac{\vec{v}_p}{|\vec{v}_p|} \right\}$ is an ***orthonormal basis*** for the span of β.

Example

Let's find an orthonormal basis for the column space of $A = \begin{array}{c} \begin{matrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{matrix} \\ \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix} \end{array}$ and find the coordinates of $\vec{y} = [5, -10, -6, 7]^T$ in terms of that basis.

orthogonal basis: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

Arbitrarily ... let $\boxed{\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}.}$

Let $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1} \vec{x}_2$

$= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} - \dfrac{-15}{30} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}$

$\boxed{= \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix}}$ $\vec{v}_2$ (check: $\vec{v}_1 \cdot \vec{v}_2 = 0$ ✓)

$\vec{v}_3 = \vec{x}_3 - \dfrac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$

$= \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix} - \dfrac{-75}{30} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \dfrac{45/2}{45/2} \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Aha!

$\vec{x}_3$ is linearly dependent on $\{\vec{x}_1, \vec{x}_2\}$.

$\vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix}$, $\vec{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{v}_4 = \vec{x}_4 - \frac{\vec{x}_4 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_4 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 - \frac{\vec{x}_4 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3$$

$\vec{0}$ vector, so its irrelevant here

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{5}{30}\begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \frac{3/2}{45/2}\begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 1/3 \\ 2/5 \\ -2/15 \end{bmatrix}$$

check: $\vec{v}_1 \cdot \vec{v}_4 = \frac{-8}{3} + \frac{4}{3} + \frac{6}{5} + \frac{2}{15} = 0$ ✓

$\vec{v}_2 \cdot \vec{v}_4 = 0 + 0 + \frac{3}{5} - \frac{9}{15} = 0$ ✓

$\therefore$ An orthogonal basis for col(A) is $\left\{ \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix}, \begin{bmatrix} 4/3 \\ 1/3 \\ 2/5 \\ -2/15 \end{bmatrix} \right\}$

An orthogonal basis that's <u>much easier</u> to work with is

$$\left\{ \vec{v}_1, \tfrac{2}{3}\vec{v}_2, 15\vec{v}_4 \right\} = \left\{ \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 20 \\ 5 \\ 6 \\ -2 \end{bmatrix} \right\}$$

An <u>orthonormal</u> basis is $\left\{ \underbrace{\frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}}_{\vec{u}_2}, \underbrace{\frac{1}{\sqrt{465}}\begin{bmatrix} 20 \\ 5 \\ 6 \\ -2 \end{bmatrix}}_{\vec{u}_3} \right\}$

note: $|\vec{u}_i| = 1 \Rightarrow \vec{u}_i \cdot \vec{u}_i = 1$

$$\Rightarrow \frac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} = \vec{y} \cdot \vec{u}_i$$

For $\vec{y} = \vec{x}_3 = \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix}$, $c_1 = \vec{y} \cdot \vec{u}_1 = \frac{-75}{\sqrt{30}}$ $\quad c_2 = \vec{y} \cdot \vec{u}_2 = \frac{15}{\sqrt{10}}$ $\quad c_3 = \vec{y} \cdot \vec{u}_3 = 0$

Check: $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = -\frac{75}{30}\begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} + \frac{15}{10}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix}$$ ✓

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

A definition and a trio of theorems

If W is a subspace of $\mathbb{R}^n$, then the ***orthogonal compliment*** of W is the set of vectors, $W^\perp$, that are orthogonal to every vector in W.

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- $(\mathrm{row}(A))^\perp = \mathrm{nul}(A)$.
- $(\mathrm{col}(A))^\perp = \mathrm{nul}(A^T)$.

ying and yang

$\mathrm{col}(A) = \mathrm{row}(A^T)$, so $(\mathrm{col}(A))^\perp = (\mathrm{row}(A^T))^\perp = \mathrm{nul}(A^T)$ ✓

Example

Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Suppose that $\vec{x} \in W^\perp$ and $\vec{y} \in W^\perp$ and $k \in \mathbb{R}$.

Then, $\forall\, \vec{w} \in W$, $\vec{x} \cdot \vec{w} = 0$ and $\vec{y} \cdot \vec{w} = 0$. So...

$\forall\, \vec{w} \in W$, $(\vec{x} + \vec{y}) \cdot \vec{w} = \vec{x} \cdot \vec{w} + \vec{y} \cdot \vec{w} = 0 + 0 = 0$ $\therefore\ \vec{x} + \vec{y} \in W^\perp$

and $\forall\, \vec{w} \in W$, $k\vec{x} \cdot \vec{w} = k \cdot (\vec{x} \cdot \vec{w}) = k \cdot 0 = 0$ $\therefore\ k\vec{x} \in W^\perp$

QED

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate that $(\text{row}(A))^{\perp} = \text{nul}(A)$.

If $\vec{x} \in \text{nul}(A)$, then $A\vec{x} = \vec{0} \implies ax_1 + bx_2 = 0$ and $cx_1 + dx_2 = 0$

If $\vec{y} \in \text{row}(A)$, the $\exists\, k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$ such that

$\vec{y} = k_1 \begin{bmatrix} a \\ b \end{bmatrix} + k_2 \begin{bmatrix} c \\ d \end{bmatrix}$ $(k_1 \text{Row}_1^T + k_2 \text{Row}_2^T)$

I need to show that vectors from row(A) & nul(A) are mutually orthogonal; i.e. $\vec{x} \cdot \vec{y} = 0$.

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix}$$
$$= x_1(k_1 a + k_2 c) + x_2(k_1 b + k_2 d)$$
$$= k_1(ax_1 + bx_2) + k_2(cx_1 + dx_2)$$
$$= k_1(0) + k_2(0)$$
$$= 0$$

QED

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate that $(\text{col}(A))^{\perp} = \text{nul}(A^{T})$.

ibid

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *unique* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$, such that $\vec{y} = \vec{w} + \vec{z}$.

In fact, if the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then

$$\vec{w} = \text{proj}_{\vec{u}_1}(\vec{y}) + \text{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \text{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \vec{z} = \vec{y} - \vec{w}$$ (comes from $\vec{y} = \vec{w} + \vec{z}$)

Example

Let $\vec{u}_1 = [1 \quad -1 \quad -1 \quad 1]^{\mathrm{T}}$, $\vec{u}_2 = [2 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}$, and $\vec{u}_3 = [-1 \quad 1 \quad 1 \quad 3]^{\mathrm{T}}$. Let $W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = [2 \quad 1 \quad 3 \quad 4]$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^{\perp}$.

$$\vec{w} = \text{proj}_{\vec{u}_1}\vec{y} + \text{proj}_{\vec{u}_2}\vec{y} + \text{proj}_{\vec{u}_3}\vec{y}$$

$$= \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\cdot\vec{u}_3$$

$$= \frac{2}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} + \frac{8}{6}\begin{bmatrix}2\\1\\1\\0\end{bmatrix} + \frac{14}{12}\begin{bmatrix}-1\\1\\1\\3\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\\2\\4\end{bmatrix}$$

$$\vec{z} = \vec{y} - \vec{w}$$

$$= \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$$

Check: $\vec{w} \in W$, $\vec{z} \in W^{\perp}$

$\vec{w}\cdot\vec{z} = 0 - 2 + 2 + 0 = 0$ ✓

$\vec{w} + \vec{z} = \begin{bmatrix}2+0\\2+(-1)\\2+1\\4+0\end{bmatrix} = \begin{bmatrix}2\\1\\3\\4\end{bmatrix} = \vec{y}$ ✓

Example

A basis from $\mathbb{R}^3$ for the $x_1 x_2 - \text{plane}$ is $\{\vec{u}_1, \vec{u}_2\}$ where $\vec{u}_1 = [3 \quad -4 \quad 0]^{\mathrm{T}}$ and $\vec{u}_2 = [4 \quad 3 \quad 0]^{\mathrm{T}}$. Use this basis to determine the point on the $x_1 x_2 - \text{plane}$ that is closest to the point $(2, 7, 5)$.

Theorem

The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.

Example

Let $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$. Discuss why $U^{\mathrm{T}}U$ must equal I_2; i.e., discuss what two phenomena (other than dimensional analysis) create that outcome.

Let $\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find an orthonormal basis for $\operatorname{span}\{\vec{x}_1, \vec{x}_2\}$. Next, express $\vec{x}_1$ and $\vec{x}_2$ in terms of that new basis.

Let's Gramm-Schmidt!

Define $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$.

Define
$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{25}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 1 \\ 8/5 \end{bmatrix}$$

(check: $\vec{v}_1 \cdot \vec{v}_2 = 0$ ✓)

∴ An orthogonal basis for span $\{\vec{x}_1, \vec{x}_2\}$ is $\left\{\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}\right\}$

∴ An orthonormal basis for span $\{\vec{x}_1, \vec{x}_2\}$ is:

$$\left\{\begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}, \begin{bmatrix} 6/\sqrt{125} \\ 5/\sqrt{125} \\ 8/\sqrt{125} \end{bmatrix}\right\} \Leftarrow \text{redefine as } \{\vec{v}_1, \vec{v}_2\}$$

Since we have an orthonormal basis, we can just use the formulas $\vec{x} \cdot \vec{v}_1$ and $\vec{x} \cdot \vec{v}_2$ to find the coordinates of $\vec{x}$ relative to $\vec{v}_1$ and $\vec{v}_2$

$$\vec{x}_1 \cdot \vec{v}_1 = \tfrac{1}{5}(4^2 + 0^2 + (-3)^2) = 5$$
$$\vec{x}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{125}}(4 \cdot 6 + 0 \cdot 5 + (-3) \cdot 8) = 0$$

$$\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix} + 0\begin{bmatrix} 6/\sqrt{125} \\ 5/\sqrt{125} \\ 8/\sqrt{125} \end{bmatrix}$$ ✓

$$\vec{x}_2 \cdot \vec{v}_1 = \frac{1}{5}(4 \cdot 2 + 0 \cdot 1 + (-3) \cdot 1) = 1$$
$$\vec{x}_2 \cdot \vec{v}_2 = \frac{1}{\sqrt{125}}(6 \cdot 2 + 5 \cdot 1 + 8 \cdot 1) = \frac{25}{\sqrt{125}}$$

$$1\begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix} + \frac{25}{\sqrt{125}}\begin{bmatrix} 6/\sqrt{125} \\ 5/\sqrt{125} \\ 8/\sqrt{125} \end{bmatrix} = \begin{bmatrix} 4/5 + 150/125 \\ 0 + 125/125 \\ -3/5 + 200/125 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \vec{x}_2$$ ✓