Projection vectors in $\mathbb{R}^n$

The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\operatorname{proj}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y} - \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The quantity $\operatorname{comp}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

Example

Find and illustrate the projection vector, component vector, and scalar component when $\vec{y} = [2,-4]^{\mathrm{T}}$ and $\vec{u} = [5,1]^{\mathrm{T}}$.

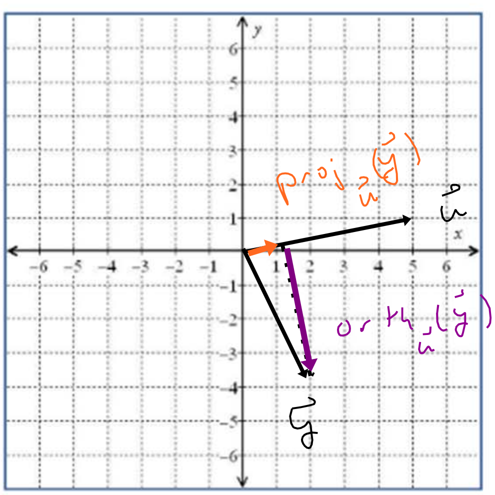
Scalar component: $\operatorname{comp}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} = \dfrac{10-4}{25+1} = 3/13$

orthogonal projection: $\operatorname{proj}_{\vec{u}}(\vec{y}) = \operatorname{comp}_{\vec{u}}(\vec{y})\,\vec{u} = \dfrac{3}{13}\cdot\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}15/13\\3/13\end{bmatrix}$

Component of $\vec{y}$ orthogonal to $\vec{u}$: $\operatorname{orth}_{\vec{u}}(\vec{y}) = y - \operatorname{proj}_{\vec{u}}\vec{y}$

$$= \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}15/13\\3/13\end{bmatrix}$$

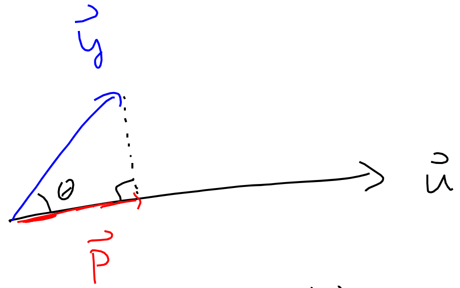
$$= \begin{bmatrix}11/13\\-55/13\end{bmatrix}$$

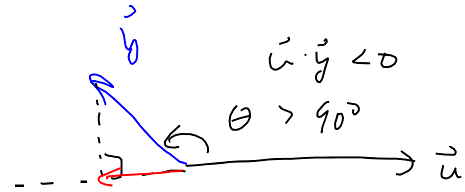


$\operatorname{proj}_{\vec{u}}(\vec{y}) = \dfrac{3}{13}\vec{u}$

$\dfrac{3}{13}$ is the component of $\vec{y}$ relative to $\vec{u}$.

$\operatorname{orth}_{\vec{u}}(\vec{y}) = \begin{bmatrix}11/13\\-55/13\end{bmatrix}$





$$\cos(\theta) = \frac{|\vec{p}|}{|\vec{y}|}$$

$$|\vec{p}| = |\vec{y}| \cos(\theta) \cdot \frac{|\vec{u}|}{|\vec{u}|}$$

$$= \frac{|\vec{y}||\vec{u}| \cos(\theta)}{|\vec{u}|}$$

$$= \frac{\vec{y} \cdot \vec{u}}{|\vec{u}|}$$

signed

Unit vector parallel to $\vec{u}$

$$\vec{p} = \text{magnitude} \cdot \frac{\vec{u}}{|\vec{u}|}$$

$$= \frac{\vec{y} \cdot \vec{u}}{|\vec{u}|} \cdot \frac{\vec{u}}{|\vec{u}|}$$

$$= \underbrace{\frac{\vec{y} \cdot \vec{u}}{|\vec{u}||\vec{u}|}}_{\text{Component of the projection}} \cdot \vec{u}$$

A definition and a theorem about orthogonal sets of vectors in $\mathbb{R}^n$

A set of non-zero vectors, $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i \cdot \vec{u}_j = 0 \;\forall\; i \neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Change of coordinates in $\mathbb{R}^n$

Let $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_\beta = [c_1, c_2, \ldots, c_n]^T$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

Example

Show that the set $\beta = \left\{ \underset{\vec{u}_1}{[1,1,1]^T}, \underset{\vec{u}_2}{[-1,-1,2]^T}, \underset{\vec{u}_3}{[3,-3,0]^T} \right\}$ is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. Then find $[\vec{y}]_\beta$ where $\vec{y} = [5,-2,0]^T$.

$\vec{u}_1 \cdot \vec{u}_2 = -1 - 1 + 2 = 0$ $\quad$ $\vec{u}_1 \cdot \vec{u}_3 = 3 - 3 + 0 = 0$ $\quad$ $\vec{u}_2 \cdot \vec{u}_3 = -3 + 3 + 0 = 0$

$\therefore$ β is indeed an orthogonal set

$$\text{Comp}_{\vec{u}_1}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{3}{3} = 1$$

$$\text{Comp}_{\vec{u}_2}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-3}{6} = -\frac{1}{2}$$

$$\text{Comp}_{\vec{u}_3}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{21}{18} = 7/6$$

Check: $1 \cdot \vec{u}_1 + \left(-\frac{1}{2}\right)\vec{u}_2 + \frac{7}{6}\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix} + \begin{bmatrix} 7/2 \\ -7/2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ ✓

$= \vec{y}$

This trivial process only works when the basis is orthogonal

The Gramm-Schmidt process **for finding an orthogonal basis**

Let $\beta = \left\{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \right\}$ be a linearly independent set of vectors.

Then the set $\left\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \right\}$ is an orthogonal basis for the span of β where:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \text{proj}_{\vec{v}_1}\left(\vec{x}_2\right) \\ \vec{v}_3 &= \vec{x}_3 - \text{proj}_{\vec{v}_1}\left(\vec{x}_3\right) - \text{proj}_{\vec{v}_2}\left(\vec{x}_3\right) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \text{proj}_{\vec{v}_1}\left(\vec{x}_p\right) - \text{proj}_{\vec{v}_2}\left(\vec{x}_p\right) - \cdots - \text{proj}_{\vec{v}_{p-1}}\left(\vec{x}_p\right) \end{aligned}$$

Furthermore, the set $\gamma = \left\{ \frac{\vec{v}_1}{\left|\vec{v}_1\right|}, \frac{\vec{v}_2}{\left|\vec{v}_2\right|}, \ldots, \frac{\vec{v}_p}{\left|\vec{v}_p\right|} \right\}$ is an ***orthonormal basis*** for the span of β.

Example

Let's find an orthonormal basis for the column space of $A = \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix}$ and find the coordinates of $\vec{y} = [5, -10, -6, 7]^{\mathrm{T}}$ in terms of that basis.

$A \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore$ $\underline{A}$ basis for col(A) is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

where $\vec{u}_1 = \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Let's Gramm-Schmidt!

$\vec{v}_1 = \vec{u}_1 = [-2, 4, 3, -1]^T$

$\vec{v}_2 = \vec{u}_2 - \text{proj}_{\vec{v}_1}(\vec{u}_2) = [1, -2, 0, 5]^T - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}[-2, 4, 3, -1]^T$

$= (1, -2, 0, 5]^T - \frac{-15}{30}[-2, 4, 3, -1]^T$

$= [0, 0, 3/2, 9/2]^T$

Let's redefine $\vec{v}_2$ to be $[0, 0, 3, 9]^T$

$\vec{v}_1 = [-2, 4, 3, -1]^T$

$\vec{v}_2 = [0, 0, 3, 9]^T$

$\vec{v}_3 = \vec{u}_3 - \text{proj}_{\vec{v}_1}(\vec{u}_3) - \text{proj}_{\vec{v}_2}(\vec{u}_3)$

$= [1,1,1,0]^T - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}[-2,4,3,-1]^T - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}[0,0,3,9]^T$

$= [1,1,1,0]^T - \frac{5}{30}[-2,4,3,-1]^T - \frac{3}{90}[0,0,3,9]^T$

$= [4/3, 1/3, 2/5, -2/15]$ Redefine $\vec{v}_3$ as $15\vec{v}_3 = [20, 5, 6, -2]^T$

Orthogonal basis: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1 = [-2,4,3,-1]^T$, $\vec{v}_2 = [0,0,1,3]^T$ & $\vec{v}_3 = [20,5,6,-2]^T$

check pair-wise $\vec{u}_i \cdot \vec{u}_j = 0$ ✓

Orthonormal basis: $\left\{ \frac{\vec{v}_1}{|\vec{v}_1|}, \frac{\vec{v}_2}{|\vec{v}_2|}, \frac{\vec{v}_3}{|\vec{v}_3|} \right\}$

$\hat{v}_1 = \frac{1}{\sqrt{30}}[-2,4,3,-1]^T$, $\hat{v}_2 = \frac{1}{\sqrt{10}}[0,0,1,3]$, $\hat{v}_3 = \frac{1}{\sqrt{465}}[20,5,6,-2]$

> **Theorem**
>
> The $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Example

Let $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$. Discuss why $U^T U$ must equal I_2; i.e., discuss what two phenomena (other than dimensional analysis) create that outcome.

Because $|\hat{v}_i| = 1$, $\frac{\vec{y} \cdot \hat{v}_i}{\hat{v}_i \cdot \hat{v}_i} = \frac{\vec{y} \cdot \hat{v}_i}{|\hat{v}_i|^2} = \vec{y} \cdot \hat{v}_i$

$\vec{y} = [5, -10, -6, 7]^T$

$\text{comp}_{\hat{v}_1}\vec{y} = \vec{y} \cdot \hat{v}_1 = \frac{-75}{\sqrt{30}}$, $\text{comp}_{\hat{v}_2}\vec{y} = \vec{y} \cdot \hat{v}_2 = \frac{15}{\sqrt{10}}$

$\text{Comp}_{\hat{v}_3}\vec{y} = \vec{y} \cdot \hat{v}_3 = 0$

Check: $(\text{comp}_{\hat{v}_1}\vec{y})\hat{v}_1 + (\text{comp}_{\hat{v}_2}\vec{y})\hat{v}_2 + (\text{comp}_{\hat{v}_3}\vec{y})\hat{v}_3$

$= -\frac{75}{\sqrt{30}} \cdot \frac{1}{\sqrt{30}}[-2,4,3,-1]^T + \frac{15}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}}[0,0,1,3]^T$

$+ 0 \cdot \frac{1}{\sqrt{465}}[20,5,6,-2] = [5,-10,-6,7]$ ✓

A definition and a trio of theorems

If W is a subspace of $\mathbb{R}^n$, then the ***orthogonal compliment*** of W is the set of vectors, $W^{\perp}$, that are orthogonal to every vector in W.

- $W^{\perp}$ is a subspace of $\mathbb{R}^n$.
- $(\text{row}(A))^{\perp} = \text{nul}(A)$.
- $(\text{col}(A))^{\perp} = \text{nul}(A^{\text{T}})$.

Example

Prove that $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Let $\vec{x}_1 \in W^{\perp}$ and $\vec{x}_2 \in W^{\perp}$. Then, $\forall \vec{w} \in W$, $\vec{x}_1 \cdot \vec{w} = 0$ and $\vec{x}_2 \cdot \vec{w} = 0$. We need to show that (1) $W^{\perp}$ is closed over vector addition and (2) $W^{\perp}$ is closed over scalar multiplication.

(1) $(\vec{x}_1 + \vec{x}_2) \cdot \vec{w} = \vec{x}_1 \cdot \vec{w} + \vec{x}_2 \cdot \vec{w}$ $\therefore$ $\vec{x}_1 + \vec{x}_2 \in W^{\perp}$

$= 0 + 0$

$= 0$

(2) $(k\vec{x}_1) \cdot \vec{w} = k(\vec{x}_1 \cdot \vec{w})$ $\therefore$ $k\vec{x}_1 \in W^{\perp}$

$= k(0)$

$= 0 \quad \forall k$

QED

$C = D^{\perp} \iff C^{\perp} = D$

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate ~~why~~ that $(\text{row}(A))^{\perp} = \text{nul}(A)$.

$\vec{x} \in nul(A) \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $ax_1 + bx_2 = 0$ and $cx_1 + dx_2 = 0$

$\vec{y} \in row(A) \Rightarrow \exists\, k_1, k_2 \in \mathbb{R}$ such that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = k_1 \underbrace{\begin{bmatrix} a \\ b \end{bmatrix} + k_2 \begin{bmatrix} c \\ d \end{bmatrix}}_{\text{Rows}^T}$

We need to show that $\vec{x} \cdot \vec{y} = 0$ (i.e. every vector in $nul(A)$ is $\perp$ to every vector in $row(A)$).

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 = x_1(k_1 a + k_2 c) + x_2(k_1 b + k_2 d)$$
$$= k_1 a x_1 + k_2 c x_1 + k_1 b x_2 + k_2 d x_2$$
$$= k_1 (ax_1 + bx_2) + k_2(cx_1 + dx_2)$$
$$= k_1(0) + k_2(0)$$
$$= 0$$

QED

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate ~~why~~ that $(\text{col}(A))^{\perp} = \text{nul}(A^{T})$.

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $\vec{x} \in nul(A^T) \Rightarrow$ $ax_1 + cx_2 = 0$ and $bx_1 + dx_2 = 0$

$\vec{y} \in col(A) \Rightarrow \exists\, k_1, k_2 \in \mathbb{R}$ such that $\vec{y} = k_1 \begin{bmatrix} a \\ c \end{bmatrix} + k_2 \begin{bmatrix} b \\ d \end{bmatrix}$

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2$$
$$= x_1 [k_1 a + k_2 b] + x_2 [k_1 c + k_2 d]$$
$$= k_1 a x_1 + k_2 b x_1 + k_1 c x_2 + k_2 d x_2$$
$$= k_1(ax_1 + cx_2) + k_2(bx_1 + dx_2)$$
$$= k_1(0) + k_2(0)$$

$$= 0$$

QED

$\star$ Check that $\vec{z} \in W^\perp$

$\vec{z} \cdot \vec{u}_1 = 0 + 1 - 1 + 0$
$= 0$ ✓

$\vec{z} \cdot \vec{u}_2 = 0 - 1 + 1 + 0$
$= 0$ ✓✓

$\vec{z} \cdot \vec{u}_3 = 0 - 1 + 1 + 0$
$= 0$ ✓✓✓

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *unique* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^\perp$, such that $\boxed{\vec{y} = \vec{w} + \vec{z}}$.

In fact, if the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then

$$\vec{w} = \text{proj}_{\vec{u}_1}(\vec{y}) + \text{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \text{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \boxed{\vec{z} = \vec{y} - \vec{w} \Rightarrow \vec{w} + \vec{z} = \vec{y}}$$

Example

Let $\vec{u}_1 = [1 \ \ -1 \ \ -1 \ \ 1]^T$, $\vec{u}_2 = [2 \ \ 1 \ \ 1 \ \ 0]^T$, and $\vec{u}_3 = [-1 \ \ 1 \ \ 1 \ \ 3]^T$. Let $W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = [2 \ \ 1 \ \ 3 \ \ 4]$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^\perp$.

Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ an orthogonal basis for W?

$\vec{u}_1 \cdot \vec{u}_2 = 2 - 1 - 1$
$= 0$

$\vec{u}_1 \cdot \vec{u}_3 = -1 - 1 - 1 + 3$
$= 0$

$\vec{u}_2 \cdot \vec{u}_3 = -2 + 1 + 1$
$= 0$

Since the vectors are pairwise orthogonal the form a linearly independent set and since $\dim(W) \le 3$ (three vector span W), $\vec{u}_1 - \vec{u}_3$ is a basis for W. $\therefore \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for W.

$$\vec{w} = \sum_{k=1}^{3} \text{proj}_{\vec{u}_k}(\vec{y})$$

$$= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 + \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}\vec{u}_3$$

$$= \frac{2}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{14}{12}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\star$

$\therefore \vec{y} = \vec{w} + \vec{z}$ where $\vec{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} \in W$ & $\vec{z} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \in W^\perp$

Example

A basis from $\mathbb{R}^3$ for the $x_1 x_2 - \textbf{plane}$ is $\{\vec{u}_1, \vec{u}_2\}$ where $\vec{u}_1 = [3 \quad -4 \quad 0]^{\mathrm{T}}$ and $\vec{u}_2 = [4 \quad 3 \quad 0]^{\mathrm{T}}$. Use this basis to determine the point on the $x_1 x_2 - \textbf{plane}$ that is closest to the point $(2, 7, 5)$.

note that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogal basis.

Lets let $\vec{y} = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$. Then

$$\vec{w} = \text{proj}_{\vec{u}_1} \vec{y} + \text{proj}_{\vec{u}_2} \vec{y}$$

$$= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

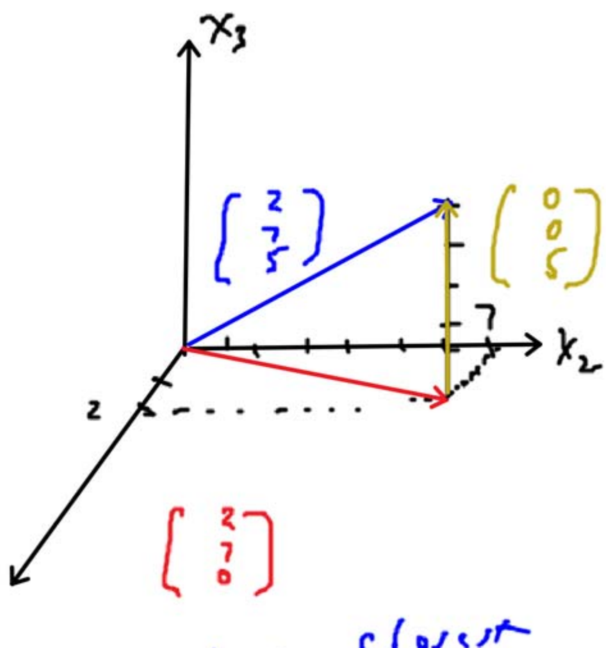
$$= \frac{-22}{25} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} + \frac{29}{25} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \vec{w}$$

$$= \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$



The point on the $x_1 x_2$-plane closest to $\begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$. Duh!