Projection vectors in $\mathbb{R}^n$

The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\text{proj}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y} - \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The quantity $\text{comp}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

Example

Find and illustrate the projection vector, component vector, and scalar component when $\vec{y} = [2,-4]^{T}$ and $\vec{u} = [5,1]^{T}$.

orthogonal projection: $\text{proj}_{\vec{u}}\vec{y} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$

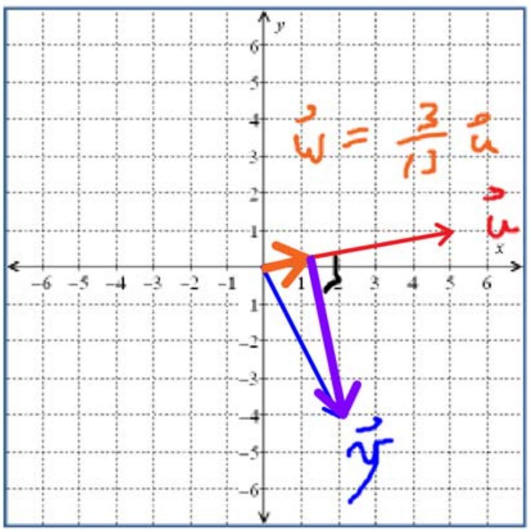
$$= \frac{6}{26}\vec{u}$$

$$= \frac{3}{13}\vec{u}$$

The component of $\vec{y}$ orthogonal to $\vec{u}$ is:

$$\vec{y} - \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \begin{bmatrix}2\\-4\end{bmatrix} - \frac{3}{13}\begin{bmatrix}5\\1\end{bmatrix}$$

$$= \begin{bmatrix}11/13\\-55/13\end{bmatrix}$$



$\vec{w} = \frac{3}{13}\vec{u} \Rightarrow \text{comp}_{\vec{u}}(\vec{y}) = \frac{3}{13}$

Let $\vec{w} = \text{proj}_{\vec{u}}(\vec{y})$

A definition and a theorem about orthogonal sets of vectors in $\mathbb{R}^n$

A set of non-zero vectors, $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i \cdot \vec{u}_j = 0 \ \forall \ i \neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Change of coordinates in $\mathbb{R}^n$

Let $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_n \vec{u}_n$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_\beta = [c_1, c_2, \ldots, c_n]^T$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

Example

Show that the set $\beta = \left\{ \underset{\vec{u}_1}{[1,1,1]^T}, \underset{\vec{u}_2}{[-1,-1,2]^T}, \underset{\vec{u}_3}{[3,-3,0]^T} \right\}$ is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. Then find $[\vec{y}]_\beta$ where $\vec{y} = [5,-2,0]^T$.

$$\text{Comp}_{\vec{u}_1}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{3}{3} = 1$$

$$\text{Comp}_{\vec{u}_2}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-3}{6} = -\frac{1}{2}$$

$$\text{Comp}_{\vec{u}_3}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{21}{18} = 7/6$$

$$\therefore [\vec{y}]_\beta = \begin{bmatrix} 1 \\ -1/2 \\ 7/6 \end{bmatrix}$$

Check $$(1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \left(\frac{7}{6}\right)\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} = \vec{y} \ \checkmark$$

The Gramm-Schmidt process **for finding an orthogonal basis**

Let $\beta = \left\{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \right\}$ be a linearly independent set of vectors.

Then the set $\left\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \right\}$ is an orthogonal basis for the span of β where:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \mathbf{proj}_{\vec{v}_1}\left(\vec{x}_2\right) \\ \vec{v}_3 &= \vec{x}_3 - \mathbf{proj}_{\vec{v}_1}\left(\vec{x}_3\right) - \mathbf{proj}_{\vec{v}_2}\left(\vec{x}_3\right) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \mathbf{proj}_{\vec{v}_1}\left(\vec{x}_p\right) - \mathbf{proj}_{\vec{v}_2}\left(\vec{x}_p\right) - \cdots - \mathbf{proj}_{\vec{v}_{p-1}}\left(\vec{x}_p\right) \end{aligned}$$

Furthermore, the set $\gamma = \left\{ \dfrac{\vec{v}_1}{\left|\vec{v}_1\right|}, \dfrac{\vec{v}_2}{\left|\vec{v}_2\right|}, \ldots, \dfrac{\vec{v}_p}{\left|\vec{v}_p\right|} \right\}$ is an ***orthonormal basis*** for the span of β.

Example

Let's find an orthonormal basis for the column space of $A = \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix}$ and find the coordinates of $\vec{y} = \left[5, -10, -6, 7\right]^{\mathrm{T}}$ in terms of that basis.

$$\begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore \text{Basis (col)} = \left\{ \vec{C}_1, \vec{C}_2, \vec{C}_4 \right\}.$$

$$\vec{v}_1 = \vec{C}_1$$

$$\vec{v}_2 = \vec{C}_2 - \frac{\vec{C}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} - \frac{-15}{30} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix}$$

$$\vec{v}_3 = \vec{c}_4 - \frac{\vec{c}_4\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{c}_4\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{5}{30}\begin{bmatrix}-2\\4\\3\\-1\end{bmatrix} - \frac{3/2}{45/2}\begin{bmatrix}0\\0\\3/2\\9/2\end{bmatrix}$$

$$= \begin{bmatrix}4/3\\1/3\\2/15\\-2/15\end{bmatrix}$$

Check: $-\frac{5\sqrt{30}}{2}\left[\frac{1}{\sqrt{30}}\begin{bmatrix}-2\\4\\3\\-1\end{bmatrix}\right] + \left(\frac{3\sqrt{10}}{2}\right)\left(\frac{1}{\sqrt{10}}\begin{bmatrix}0\\0\\1\\3\end{bmatrix}\right) + 0\left(\frac{1}{\sqrt{465}}\begin{bmatrix}20\\5\\6\\-2\end{bmatrix}\right)$

$= \begin{bmatrix}5\\-10\\-6\\7\end{bmatrix}$

$\therefore$ An orthogonal Basis for Col (A) is $\left\{\begin{bmatrix}-2\\4\\3\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\3\end{bmatrix}, \begin{bmatrix}20\\5\\6\\-2\end{bmatrix}\right\}$

mental check orthogonality

$\therefore$ An orthonormal Basis for col(A) is $\left\{\frac{1}{\sqrt{30}}\begin{bmatrix}-2\\4\\3\\1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}0\\0\\1\\3\end{bmatrix}, \frac{1}{\sqrt{465}}\begin{bmatrix}20\\5\\6\\-2\end{bmatrix}\right\} = \gamma$

$\vec{y} = \begin{bmatrix}5\\-10\\-6\\7\end{bmatrix}$; $[\vec{y}]_\gamma = \begin{bmatrix}\frac{1}{\sqrt{30}}(-75)\\ \frac{1}{\sqrt{10}}(15)\\ \frac{1}{\sqrt{465}}(0)\end{bmatrix} = \begin{bmatrix}-\frac{5\sqrt{30}}{2}\\ \frac{3\sqrt{10}}{2}\\ 0\end{bmatrix}$

$\therefore$

> **Theorem**
>
> The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.

$\vec{u}_i \cdot \vec{u}_j = 0$ $\quad$ $|\vec{u}| = 1$

Example

Let $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} \\ -1/\sqrt{3} & 3/\sqrt{14} \end{bmatrix}$. Discuss why $U^{\mathrm{T}}U$ must equal I_2; i.e., discuss what two phenomena (other than dimensional analysis) create that outcome.

$$\begin{array}{cc} \vec{c}_1 & \vec{c}_2 \end{array}$$

$$u^T u = \underbrace{\begin{bmatrix}\vec{c}_1\\ \vec{c}_2\end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix}\vec{c}_1 & \vec{c}_2\end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix}a & b\\ c & d\end{bmatrix}}_{2\times 2}$$

$$\vec{c}_1\cdot\vec{c}_1 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1 \Rightarrow |\vec{c}_1| = 1$$

$$\vec{c}_2\cdot\vec{c}_2 = \frac{1}{14}+\frac{4}{14}+\frac{9}{14} = 1 \Rightarrow |\vec{c}_2| = 1$$

$$\vec{c}_1\cdot\vec{c}_2 = \frac{1}{\sqrt{42}}+\frac{2}{\sqrt{42}}-\frac{3}{\sqrt{42}} = 0 \Rightarrow$$ $\vec{c}_1$ is orthogonal with $\vec{c}_2$

$$U^T U = \begin{bmatrix} \vec{c}_1 \\ \vec{c}_2 \end{bmatrix} [\vec{c}_1 \ \vec{c}_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a = \vec{c}_1 \cdot \vec{c}_1 = 1$$
$$b = \vec{c}_1 \cdot \vec{c}_2 = 0$$
$$c = \vec{c}_2 \cdot \vec{c}_1 = 0$$
$$d = \vec{c}_2 \cdot \vec{c}_2 = 1$$

$$\therefore U^T U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A definition and a trio of theorems

If W is a subspace of $\mathbb{R}^n$, then the ***orthogonal compliment*** of W is the set of vectors, $W^\perp$, that are orthogonal to every vector in W.

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- For a square matrix A, $(\text{row}(A))^\perp = \textbf{nul}(A)$.
- For a square matrix A, $(\text{col}(A))^\perp = \textbf{nul}(A^T)$.

Example

Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Let $\vec{w} \in W$ and $\vec{u} \in W^\perp$ and $\vec{v} \in W^\perp$

Then $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$. So...

1) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
$= 0 + 0$
$= 0$

2) $K\vec{u} \cdot \vec{w} = K(\vec{u} \cdot \vec{w})$
$= K(0)$
$= 0$ QED

$\vec{w} \in V$ and $\vec{x} \in V^\perp$
means $\vec{w} \cdot \vec{x} = 0$

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate why $(\text{row}(A))^\perp = \text{nul}(A)$.

Let $\vec{u} \in \text{row}(A)$ and $\vec{w} \in \text{nul}(A)$. We need to show that $\vec{u} \cdot \vec{w} = 0$.

$\exists\, c_1, c_2 \in \mathbb{R} \ni \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix}$

$\vec{w} \in \text{nul}(A) \Rightarrow \begin{matrix} aw_1 + bw_2 = 0 \\ cw_1 + dw_2 = 0 \end{matrix}$

$$\begin{aligned} \vec{u} \cdot \vec{w} &= u_1 w_1 + u_2 w_2 \\ &= (c_1 a + c_2 c) w_1 + (c_1 b + c_2 d) w_2 \\ &= c_1 a w_1 + c_1 b w_2 + c_2 c w_1 + c_2 d w_2 \\ &= c_1 (a w_1 + b w_2) + c_2 (c w_1 + d w_2) \\ &= c_1 (0) + c_2 (0) = 0 \quad \text{QED} \end{aligned}$$

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate why $(\text{col}(A))^\perp = \text{nul}(A^T)$.

Let $\vec{u} \in \text{col}(A)$ and $\vec{v} \in \text{nul}(A^T)$. We gotta show that $\vec{u} \cdot \vec{v} = 0$.

$\exists\, c_1, c_2 \in \mathbb{R} \ni \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 \begin{bmatrix} a \\ c \end{bmatrix} + c_2 \begin{bmatrix} b \\ d \end{bmatrix}$

$\vec{v} \in \text{nul}(A^T) \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a v_1 + c v_2 = 0 \\ b v_1 + d v_2 = 0 \end{cases}$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 \\ &= (c_1 a + c_2 b) v_1 + (c_1 c + c_2 d) v_2 \\ &= c_1 a v_1 + c_1 c v_2 + c_2 b v_1 + c_2 d v_2 \\ &= c_1 (a v_1 + c v_2) + c_2 (b v_1 + d v_2) \\ &= c_1 (0) + c_2 (0) = 0 \quad \text{QED} \end{aligned}$$

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *unique* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$, such that $\vec{y} = \vec{w} + \vec{z}$.

In fact, if the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then

Formula 1 $\vec{w} = \text{proj}_{\vec{u}_1}(\vec{y}) + \text{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \text{proj}_{\vec{u}_p}(\vec{y})$ and $\vec{z} = \vec{y} - \vec{w}$

Example

Let $\vec{u}_1 = [1 \quad -1 \quad -1 \quad 1]^T$, $\vec{u}_2 = [2 \quad 1 \quad 1 \quad 0]^T$, and $\vec{u}_3 = [-1 \quad 1 \quad 1 \quad 3]^T$. Let $W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = [1 \quad 1 \quad 1 \quad 1]$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^{\perp}$.

Formula 1 applies iff $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis.

① $\vec{u}_1 \cdot \vec{u}_2 = 2 - 1 - 1 + 0 = 0$ $\quad$ $\vec{u}_2 \cdot \vec{u}_3 = -2 + 1 + 1 + 0 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = -1 - 1 - 1 + 3 = 0$

② There's a theorem that states that an orthogonal set is linearly independent.

③ $W = \text{span}(\{\vec{u}_1, \vec{u}_2, \vec{u}_3\})$ so clearly $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ spans W.

Good to go!

$$\vec{w} = \text{proj}_{\vec{u}_1}(\vec{y}) + \text{proj}_{\vec{u}_2}(\vec{y}) + \text{proj}_{\vec{u}_3}(\vec{y})$$

$$= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 + \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}\vec{u}_3$$

$$= \frac{0}{4}\vec{u}_1 + \frac{4}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{12}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \vec{y} \; ; \; \vec{y} \in W$$

$\vec{y} = \vec{w} + \vec{z}$ where $\vec{w} = \vec{y}$, $\vec{z} = \vec{0}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$

Example

A basis from $\mathbb{R}^3$ for the $x_1 x_2-\textbf{plane}$ is $\{\vec{u}_1, \vec{u}_2\}$ where $\vec{u}_1 = [3 \quad -4 \quad 0]^{\mathrm{T}}$ and $\vec{u}_2 = [4 \quad 3 \quad 0]^{\mathrm{T}}$. Use this basis to determine the point on the $x_1 x_2-\textbf{plane}$ that is closest to the point $(2,7,5)$.

note that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis.

Lets let $\vec{y} = \begin{bmatrix}2\\7\\5\end{bmatrix}$. Then

$$\vec{w} = \text{proj}_{\vec{u}_1}\vec{y} + \text{proj}_{\vec{u}_2}\vec{y}$$

$$= \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

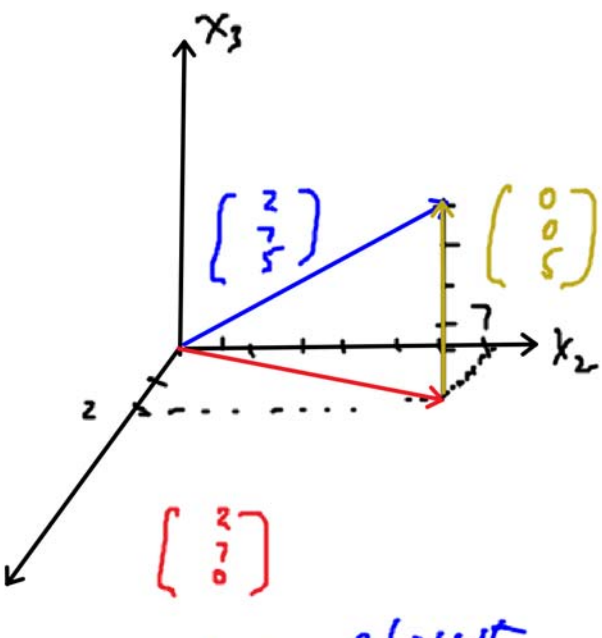
$$= \frac{-22}{25}\begin{bmatrix}3\\-4\\0\end{bmatrix} + \frac{29}{25}\begin{bmatrix}4\\3\\0\end{bmatrix}$$

$$= \begin{bmatrix}2\\7\\0\end{bmatrix}$$

$$\vec{z} = \vec{y} - \vec{w}$$

$$= \begin{bmatrix}2\\7\\5\end{bmatrix} - \begin{bmatrix}2\\7\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\5\end{bmatrix}$$



The point on the $x_1 x_2$-plane closest to $\begin{bmatrix}2\\7\\5\end{bmatrix}$ is $\begin{bmatrix}2\\7\\0\end{bmatrix}$. Duh!