2) $\text{proj}_{\vec{u}}(\vec{y})$ is $\frac{3}{13}$ as long as $\vec{u}$.

Projection vectors in $\mathbb{R}^n$

The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y} - \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The quantity $\text{comp}_{\vec{u}}(\vec{y}) = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

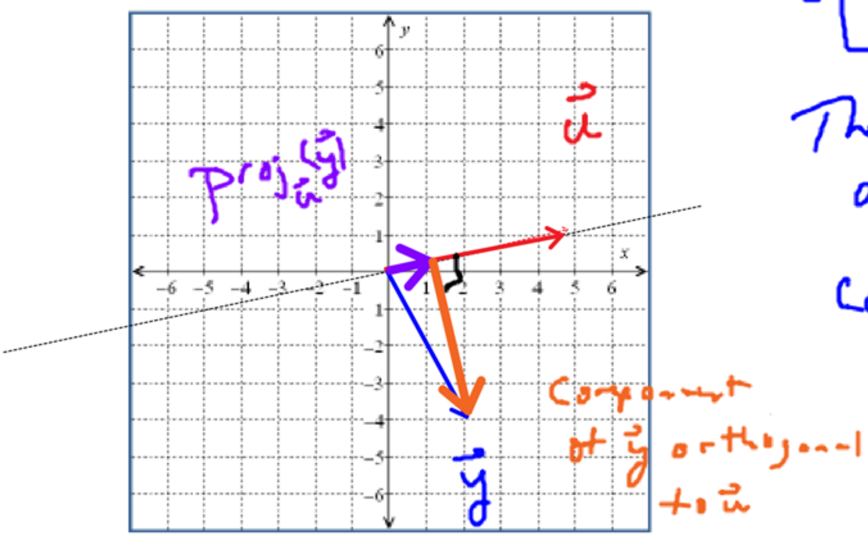
Example

Find and illustrate the projection vector, component vector, and scalar component when $\vec{y} = [2,-4]^{\mathrm{T}}$ and $\vec{u} = [5,1]^{\mathrm{T}}$.

$$\text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{6}{26}\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}15/13\\3/13\end{bmatrix}$$

The component of $\vec{y}$ orthogonal to $\vec{u}$ is

$$\vec{y} - \text{proj}_{\vec{u}}(\vec{y}) = \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}15/13\\3/13\end{bmatrix} = \begin{bmatrix}11/13\\-55/13\end{bmatrix}$$



The scalar component of $\vec{y}$ onto $\vec{u}$ is

$$\text{comp}_{\vec{u}}(\vec{y}) = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} = 3/13$$

$\text{comp}_{\vec{u}}(\vec{y}) = \frac{3}{13}$ tells us two things

1) The projection points in the same direction as $\vec{u}$ (comp > 0)

A definition and a theorem about orthogonal sets of vectors in $\mathbb{R}^n$

A set of non-zero vectors, $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i \cdot \vec{u}_j = 0 \ \forall\ i \neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Change of coordinates in $\mathbb{R}^n$

Let $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_\beta = [c_1, c_2, \ldots, c_n]^T$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

Example

Show that the set $\beta = \left\{[1,1,1]^T, [-1,-1,2]^T, [3,-3,0]^T\right\}$ is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. Then find $[\vec{y}]_\beta$ where $\vec{y} = [5,-2,0]^T$.

Define $\vec{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix}3\\-3\\0\end{bmatrix}$.

$\vec{u}_1 \cdot \vec{u}_2 = -1-1+2 = 0$ $\quad \vec{u}_1 \cdot \vec{u}_3 = 3-3+0 = 0$ $\quad \vec{u}_2 \cdot \vec{u}_3 = -3+3+0 = 0$

Thus the set is orthogonal and, consequently, also linearly indepent. A set of three linearly independent vectors from $\mathbb{R}^3$ always form a basis for $\mathbb{R}^3$.

For $\vec{y}$, $c_1 = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \dfrac{3}{3} = 1$ $\quad c_2 = \dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \dfrac{-3}{6} = -\dfrac{1}{2}$ $\quad c_3 = \dfrac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \dfrac{21}{18} = \dfrac{7}{6}$

Check: $(1)\begin{bmatrix}1\\1\\1\end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix}-1\\-1\\2\end{bmatrix} + \left(\frac{7}{6}\right)\begin{bmatrix}3\\-3\\0\end{bmatrix} = \begin{bmatrix}5\\-2\\0\end{bmatrix} = \vec{y}$ ✓

$\therefore [\vec{y}]_\beta = \begin{bmatrix}1\\-1/2\\7/6\end{bmatrix}$

<u>The Gramm-Schmidt process</u> **for finding an orthogonal basis**

Let $\beta = \left\{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \right\}$ be a linearly independent set of vectors.

Then the set $\left\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \right\}$ is an orthogonal basis for the span of β where:

$$\begin{aligned}
\vec{v}_1 &= \vec{x}_1 \\
\vec{v}_2 &= \vec{x}_2 - \mathbf{proj}_{\vec{v}_1}\left(\vec{x}_2\right) \\
\vec{v}_3 &= \vec{x}_3 - \mathbf{proj}_{\vec{v}_1}\left(\vec{x}_3\right) - \mathbf{proj}_{\vec{v}_2}\left(\vec{x}_3\right) \\
&\vdots \\
\vec{v}_p &= \vec{x}_p - \mathbf{proj}_{\vec{v}_1}\left(\vec{x}_p\right) - \mathbf{proj}_{\vec{v}_2}\left(\vec{x}_p\right) - \cdots - \mathbf{proj}_{\vec{v}_{p-1}}\left(\vec{x}_p\right)
\end{aligned}$$

Furthermore, the set $\gamma = \left\{ \dfrac{\vec{v}_1}{\left|\vec{v}_1\right|}, \dfrac{\vec{v}_2}{\left|\vec{v}_2\right|}, \ldots, \dfrac{\vec{v}_p}{\left|\vec{v}_p\right|} \right\}$ is an ***<u>orthonormal basis</u>*** for the span of β.

Example

Let's find an orthonormal basis for the column space of $A = \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix}$ and find the coordinates of $\vec{y} = \left[5, -10, -6, 7\right]^{\mathrm{T}}$ in terms of that basis.

Let $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix}$, $\vec{x}_4 = \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix}$

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$= \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \dfrac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -11/3 \\ 7/3 \\ 4/3 \\ -1 \end{bmatrix}$

<u>Simonds' adaptation</u>

redefine $\vec{v}_2 = \begin{bmatrix} -11 \\ 7 \\ 4 \\ -3 \end{bmatrix}$

Check orthogonality of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$

$\vec{v}_1 \cdot \vec{v}_2 = -11+7+4+0 = 0$ ✓ | $\vec{v}_1 \cdot \vec{v}_3 = -12-3+15+0 = 0$ ✓ | $\vec{v}_2 \cdot \vec{v}_3 = 0$ ✓

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} - \left(\frac{-1}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-40}{195}\right)\begin{bmatrix} -11 \\ 7 \\ 4 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -12/13 \\ -3/13 \\ 15/13 \\ 57/13 \end{bmatrix} \qquad \text{redefine } \vec{v}_3 = \begin{bmatrix} -12 \\ -3 \\ 15 \\ 57 \end{bmatrix}$$

$$\vec{v}_4 = \vec{x}_4 - \frac{\vec{x}_4 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_4 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 - \frac{\vec{x}_4 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3$$

$$= \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix} - \left(\frac{-11}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-170}{195}\right)\begin{bmatrix} -11 \\ 7 \\ 4 \\ -3 \end{bmatrix} - \left(\frac{279}{3627}\right)\begin{bmatrix} -12 \\ -3 \\ 15 \\ 57 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ An orthogonal basis for $\text{col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 7 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -12 \\ -3 \\ 15 \\ 57 \end{bmatrix} \right\}$$

note that this implies that the columns of A are linearly dependent.

> **Theorem**
>
> The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.

$\vec{u}_i \cdot \vec{u}_j = 0$ $\quad$ $|\vec{a}_i| = 1$

Example

Let $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{10} \\ 1/\sqrt{3} & 2/\sqrt{10} \\ -1/\sqrt{3} & 5/\sqrt{10} \end{bmatrix}$. Discuss why $U^{\mathrm{T}}U$ must equal I_2; i.e., discuss what two phenomena (other than dimensional analysis) create that outcome.

See supplemental notes

A definition and a trio of theorems

If W is a subspace of $\mathbb{R}^n$, then the ***orthogonal compliment*** of W is the set of vectors, $W^\perp$, that are orthogonal to every vector in W.

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- For a square matrix A, $(\text{row}(A))^\perp = \textbf{nul}(A)$.
- For a square matrix A, $(\text{col}(A))^\perp = \textbf{nul}(A^T)$.

Example

Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Let $\vec{u} \in W^\perp$, $\vec{v} \in W^\perp$, $k \in \mathbb{R}$ and $\vec{w} \in W$.

Closure over addition?

$\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$ (by definition of $W^\perp$)

ergo, $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
$= 0 + 0$
$= 0$

Thus, $\vec{u} + \vec{v} \in W^\perp$

closure over multiplication?

$(k\vec{u}) \cdot \vec{w} = k(\vec{u} \cdot \vec{w})$
$= k(0)$
$= 0$

Hence, $k\vec{u} \in W^\perp$

QED

Example

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate why $(\text{row}(A))^{\perp} = \text{nul}(A)$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{nul}(A)$. By definition $A\vec{x} = \vec{0}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \overrightarrow{row_1} \cdot \vec{x} = 0 \text{ and } \overrightarrow{row_2} \cdot \vec{x} = 0$$

$$\Rightarrow (k_1 \overrightarrow{row_1}) \cdot \vec{x} + (k_2 \overrightarrow{row_2}) \cdot \vec{x} = 0$$

$\Rightarrow$ every vector in row(A) is orthogonal to every vector in nul(A)

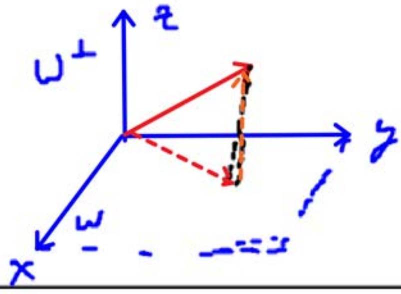
Example

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate why $(\text{col}(A))^{\perp} = \text{nul}(A^{\text{T}})$.

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ so $\vec{x} \in \text{nul}(A^T)$ gives us:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} ax_1 + cx_2 \\ bx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \overrightarrow{col_1(A)} \cdot \vec{x} = 0 \text{ and } col_2(A) \cdot \vec{x} = 0$$



The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *unique* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^\perp$, such that $\vec{y} = \vec{w} + \vec{z}$.

In fact, if the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then

$$\vec{w} = \mathbf{proj}_{\vec{u}_1}(\vec{y}) + \mathbf{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \mathbf{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \vec{z} = \vec{y} - \vec{w}$$

Example

Let $\vec{u}_1 = [1 \ \ -1 \ \ -1 \ \ 1]^{\mathrm{T}}$, $\vec{u}_2 = [2 \ \ 1 \ \ 1 \ \ 0]^{\mathrm{T}}$, and $\vec{u}_3 = [-1 \ \ 1 \ \ 1 \ \ 3]^{\mathrm{T}}$. Let $W = \operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = [1 \ \ 1 \ \ 1 \ \ 1]$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^\perp$.

$\vec{u}_1 \cdot \vec{u}_2 = 2 - 1 - 1 + 0 = 0$ $\quad$ $\vec{u}_1 \cdot \vec{u}_3 = -1 - 1 - 1 + 3 = 0$ $\quad$ $\vec{u}_2 \cdot \vec{u}_3 = -2 + 1 + 1 + 0 = 0$

$\therefore$ $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthogonal and, consequently, linearly independent. So, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

$$\begin{aligned}
\vec{w} &= \text{proj}_{\vec{u}_1}\vec{y} + \text{proj}_{\vec{u}_2}\vec{y} + \text{proj}_{\vec{u}_3}\vec{y} \\
&= \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3 \\
&= \frac{0}{4}\vec{u}_1 + \frac{4}{6}\cdot\begin{bmatrix}2\\1\\1\\0\end{bmatrix} + \frac{4}{12}\begin{bmatrix}-1\\1\\1\\3\end{bmatrix} \\
&= \begin{bmatrix}1\\1\\1\\1\end{bmatrix}
\end{aligned}$$

$\therefore \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \in W$ because $\text{proj}_W \vec{y} = \vec{y}$

$\vec{z} = \vec{y} - \vec{w} = \vec{0}$.

$\therefore \vec{y} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ where $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \in W$ and $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \in W^\perp$

Example

A basis from $\mathbb{R}^3$ for the $x_1\,x_2-\textbf{plane}$ is $\{\vec{u}_1,\vec{u}_2\}$ where $\vec{u}_1=\begin{bmatrix}3 & -4 & 0\end{bmatrix}^T$ and $\vec{u}_2=\begin{bmatrix}4 & 3 & 0\end{bmatrix}^T$. Use this basis to determine the point on the $x_1\,x_2-\textbf{plane}$ that is closest to the point $(2,7,5)$.

The point on the x_1x_2-plane that is closest to $(2,7,5)$ is $\text{proj}_V\,\vec{y}$ where $\vec{y}=\begin{bmatrix}2\\7\\5\end{bmatrix}$ and $V=\text{span}\left\{\begin{bmatrix}3\\-4\\0\end{bmatrix},\begin{bmatrix}4\\3\\0\end{bmatrix}\right\}$

Because $\left\{\begin{bmatrix}3\\-4\\0\end{bmatrix},\begin{bmatrix}4\\3\\0\end{bmatrix}\right\}$ is an orthogonal basis for x_1x_2-plane, this projection is

$\vec{w}=\text{proj}_{\vec{u}_1}\vec{y}+\text{proj}_{\vec{u}_2}\vec{y}$ where $\vec{u}_1=\begin{bmatrix}3\\-4\\0\end{bmatrix}$ and $\vec{u}_2=\begin{bmatrix}4\\3\\0\end{bmatrix}$

$$\therefore\quad \vec{w}=\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1+\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

$$=\frac{-22}{25}\begin{bmatrix}3\\-4\\0\end{bmatrix}+\frac{29}{25}\begin{bmatrix}4\\3\\0\end{bmatrix}$$

$$=\begin{bmatrix}2\\7\\0\end{bmatrix}$$