Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$.
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $m - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is ____________________.

The column space of M is a __________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The row space of M is a ____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The null space of M is a ____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\text{row}(M)$.

True or false? The stated basis for $\text{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

State a basis for $\text{col}(M)$.

True or false? The stated basis for $\text{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathrm{nul}(M)$.

True or false? The stated basis for $\mathrm{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

Example

Find bases for $\text{row}(A)$, $\text{col}(A)$, and $\text{nul}(A)$ where $A=\begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

What are bases for the kernel and range of the linear transformation $T\left(\vec{x}\right)=A\vec{x}$ where $A=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$. What does this imply must be true about A^2.

Explain ***geometrically*** why the rotation matrix $R=\begin{bmatrix}\cos\left(\theta\right) & -\sin\left(\theta\right)\\ \sin\left(\theta\right) & \cos\left(\theta\right)\end{bmatrix}$ cannot possibly have any real number eigenvalues for $0<\theta<\pi$.

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -2 \\ -2 & 2 & -8 & 0 \\ 3 & 2 & 7 & 10 \end{bmatrix}$ and note that $A \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for each of the following vector spaces ***without actually transposing the matrix*** $\boldsymbol{A}$. In each case, write a few words so that the rationale for whatever action/conclusion you take/make is clear.

a. $\text{null}\left(A^{\mathsf{T}}\right)^{\perp}$

b. $\text{col}\left(A^{\mathsf{T}}\right)^{\perp}$