Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$. This basis comes from the original matrix
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$ This basis comes from the RREF matrix.
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $m - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

$\underbrace{\text{dim (row space)}}_{\text{rank}}$ + dim (null space) = # of columns.

rank ← Also the dimension of the column space.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is 3.

The column space of M is a 3-dimensional subspace of $\mathbb{R}^{4}$.

The row space of M is a 3-dimensional subspace of $\mathbb{R}^{7}$.

The null space of M is a 4-dimensional subspace of $\mathbb{R}^{7}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathrm{row}(M)$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 5 \\ -2 \end{bmatrix}^T \right\}$$

True or false? The stated basis for $\mathrm{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

True. Row Equivalent matrices are produced via

(1) Swapping rows (contextually irrelevant)

(2) Replacing rows with linear combinations of existing rows.

State a basis for $\mathrm{col}(M)$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

True or false? The stated basis for $\mathrm{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

Nothing falser could be. To wit:

$$M: \; R_3 + R_4 \to R_4 \quad \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \end{bmatrix}$$ (call this M_{new})

$\begin{bmatrix} 5 \\ 3 \\ -2 \\ -2 \end{bmatrix} \in \mathrm{col}(M_{new})$, but obviously $\begin{bmatrix} 5 \\ 3 \\ -2 \\ -2 \end{bmatrix}$ is not is the span stated in the last question.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathrm{nul}(M)$.

"Scratch work"

$M\vec{x} = \vec{0}$

$x_1 = -2x_3 - 5x_7$

$x_2 = x_3 - 3x_7$

$x_4 = -5x_6 + 2x_7$

x_3, x_5, x_6, x_7 are free

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$x_3 = 1$ $\quad x_5 = 1$ $\quad x_6 = 1$ $\quad x_7 = 1$

True or false? The stated basis for $\mathrm{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

Nothing could be truer. This is the entire basis of this class.

Adding non-zero multiples of two sides of two equations from a system of equations does not change the solution set

$A\vec{x} = \vec{0}$

$\left[A \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$ Row reduction

Example

Find bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$, and $\operatorname{nul}(A)$ where $A = \begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ -1 \end{bmatrix}^T \right\}$ rank = 2

Basis for column space: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix} \right\}$

Basis for null space: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

dimensions add to 5 ← # of columns

What are bases for the kernel and range of the linear transformation $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

What does this imply must be true about A^2.

Null Space: $A\vec{x} = \vec{0}$, $x_1 = x_2$: Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

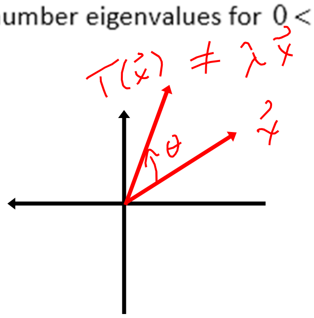
Range: $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$: Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{T} \underbrace{k \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{in the null space of } A \text{ (kernel of } T)} \xrightarrow{T} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(T(\vec{x})) = \vec{0} \Rightarrow A(A\vec{x}) = \vec{0} \quad \forall \vec{x}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Explain ***geometrically*** why the rotation matrix $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ cannot possibly have any real number eigenvalues for $0 < \theta < \pi$.



If λ was an eigenvalue for R, for some $\vec{x}$ we'd have $R\vec{x} = \lambda \vec{x}$

QED

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -2 \\ -2 & 2 & -8 & 0 \\ 3 & 2 & 7 & 10 \end{bmatrix}$ and note that $A \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Find a basis for each of the following vector spaces ***without actually transposing the matrix A***. In each case, write a few words so that the rationale for whatever action/conclusion you take/make is clear.

a. $\text{null}(A^{\mathsf{T}})^{\perp}$

$$\text{null}(A^T)^{\perp} = \text{row}(A^T)$$
$$= \text{col}(A)$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 10 \end{bmatrix} \right\}$

b. $\text{col}(A^{\mathsf{T}})^{\perp}$

$$\text{col}(A^T)^{\perp} = \text{row}(A)^{\perp}$$
$$= \text{nul}(A)$$

Basis: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$