Let $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. An orthonormal basis for $\text{col}(A)$ is $\beta = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \frac{1}{\sqrt{2}}[0,1,1]^T$ and $\vec{b}_2 = \frac{1}{\sqrt{6}}[2,-1,1]^T$.

a. Let $\vec{u} = [5,-1,4]^T$. Use dot products to determine $[\vec{u}]_\beta$ and verify the result.

$\vec{u} \cdot \vec{b}_1 = \frac{3}{\sqrt{2}}$ $\qquad$ $\vec{u} \cdot \vec{b}_2 = \frac{15}{\sqrt{6}}$

$[\vec{u}]_\beta = \begin{bmatrix} 3/\sqrt{2} \\ 15/\sqrt{6} \end{bmatrix}$ $\qquad$ $w_1 \vec{b}_1 + w_2 \vec{b}_2$

Check: $\frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{15}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 3/2 \\ 3/2 \end{bmatrix} + \begin{bmatrix} 5 \\ -5/2 \\ 5/2 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ woohoo!

b. Let $\vec{v} = [1,3,2]^T$. Use dot products to determine $[\vec{v}]_\beta$. This time the result does not verify! How can that be? What's the problem??

$\vec{v} \cdot \vec{b}_1 = \frac{5}{\sqrt{2}}$ $\qquad$ $\vec{v} \cdot \vec{b}_2 = \frac{1}{\sqrt{6}}$

Show that $\frac{5}{\sqrt{2}} \vec{b}_1 + \frac{1}{\sqrt{6}} \vec{b}_2 = \vec{v}$

$\frac{5}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \\ 5/2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/6 \\ 1/6 \end{bmatrix}$

$\neq \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ wah!

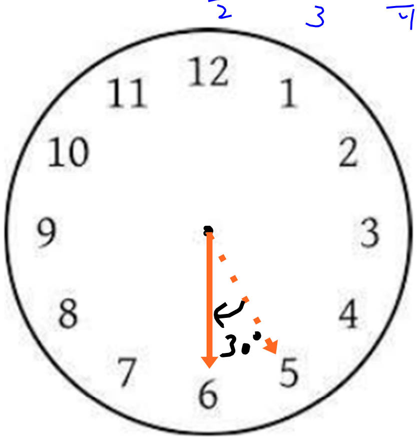
β is a basis for the column space of A. $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \notin \text{col}(A)$

Algebraic structures are generally defined by objects in a set (called elements) and at least one operation that is performed between the elements in the set. There is inevitably an identity element in the set with the property that performing the operation between the identity element and x always results in x.

An algebraic structure is called **cyclic of modulus *n*** if performing the operation between *n* occurances of any given element results in the identity element. For example, clock arithmetic is cyclic with modulus twelve (twelve being the identity element for clock arithmetic). Each element of a cyclic algebraic stucture has an order, *m*, which is the smallest number of times the operation can be performed between the element and itself before getting to the identity element. Let's enter the order for each hour on a clock into Table 1 and deduce from that a formula for the order of "hour k."

Table 1: Clock Arithmetic

hour	1	2	3	4	5	6	7	8	9	10	11	12
order	12	6	4	3	12	2	12	3	4	6	12	1
		$\frac{12}{2}$	$\frac{12}{3}$	$\frac{12}{4}$	$\frac{12}{1}$	$\frac{12}{6}$	$\frac{12}{1}$	$\frac{12}{4}$	$\frac{12}{3}$	$\frac{12}{2}$	$\frac{12}{1}$	$\frac{12}{12}$



$$m = \text{order}(k) = \frac{12}{\text{GCF}(k,12)}$$

Sometimes matrix multiplication is cyclic (with I being the identity element). For example, If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So we say that powers of A are cyclic with modulus 4.

Use something we learned about this term to find a matrix whose powers are cyclic with modulus 12.

If $A^{12} = I$, then $A^{12}\vec{x} = \vec{x}$

Rotating $\vec{x}$ by 30° twelve times results in $\vec{x}$

$$A = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underbrace{T(\vec{x}) = \vec{0}}_{\vec{x} \in \text{Kernel of } T} \iff \underbrace{A\vec{x}}_{\vec{x} \in \text{null space of } A} = \vec{0}$$

What are bases for the kernel and range of the linear transformation $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

What does this imply must be true about A^2.

$\text{Ker}(T) = \text{null}(A)$ $\quad x_1 = x_2 \quad$ Basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Range of T: Set of all images of vectors in $\mathbb{R}^2$ under T

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_2 \end{bmatrix}$$

like components

Basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} k \\ k \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in null space $\quad$ 100% of time

$$\therefore \quad A(A\vec{x}) = \underbrace{A^2\vec{x} = \vec{0}}_{100\% \text{ of the time}} \qquad \therefore \; A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Explain ***geometrically*** why the rotation matrix $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ cannot possibly have any real number eigenvalues for $0 < \theta < \pi$.

R rotates $\vec{x}$ by θ where $0^\circ < \theta < 180^\circ$.

If λ were a real # eigenvalue of R, then

$R\vec{x} = \lambda\vec{x} \Rightarrow$ $R\vec{x}$ points in the same or opposite direction as $\vec{x}$

$\text{row}(B)^{\perp} = \text{nul}(B)$ $\qquad$ $\text{nul}(B)^{\perp} = \text{row}(B)$

MTH 261 – Mr. Simonds' class

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -2 \\ -2 & 2 & -8 & 0 \\ 3 & 2 & 7 & 10 \end{bmatrix}$ and note that $A \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for each of the following vector spaces ***without actually transposing the matrix A***. In each case, write a few words so that the rationale for whatever action/conclusion you take/make is clear.

a. $\text{null}\left(A^{T}\right)^{\perp}$

$$\text{null}(A^T)^{\perp} = \text{row}(A^T) = \text{col}(A)$$

$$\text{Basis: } \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 10 \end{bmatrix} \right\}$$

b. $\text{col}\left(A^{T}\right)^{\perp}$

$$\text{Col}(A^T)^{\perp} = \text{row}(A)^{\perp} = \text{nul}(A)$$

$A\vec{x} = \vec{0}$ $\quad$ gen solution: $\begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$

$$\text{Basis: } \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$