Example

Pretend that Manhattan only has Midtown, the Upper East Side, and the Upper West Side. Suppose that cabbie shifts change only at 6 am and 6 pm. Suppose that a state vector for the distribution of cabs has form $[\text{Midtown}, \text{UES}, \text{UWS}]^T$ and that the transition matrix over each 5 minute interval between 6 am and 6 pm is the matrix P. Find the steady state vector for this model. Suppose that the cabs are evenly distributed at the start of the 6 am shift; how well does the steady state vector predict the distribution of the cabs at the end of that shift?

$$P = \begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}$$

The eigenvalues for P are nasty.

Let's go with plan B.

Let $\vec{v}$ be the steady state vector. Then

$$P\vec{v} = \vec{v}.$$

So $\vec{v}$ is the stochastic solution to

$$(P - 1I)\vec{v} = \vec{0}$$

Geez, plan B is a whole lot less work than plan A in all cases!

So plan A ⟷ plan B

$$\left[\begin{array}{ccc|c} -.5 & .3 & .4 & 0 \\ .2 & -.4 & .1 & 0 \\ .3 & .1 & -.5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -19/14 & 0 \\ 0 & 1 & -13/14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Clearly $x_1 = \frac{19}{14}x_3$ and $x_2 = \frac{13}{14}x_3$, so one solution to $P\vec{v} = \vec{v}$ is $\begin{bmatrix} 19 \\ 13 \\ 14 \end{bmatrix}$

But this ain't the steady state vector because the entries don't sum to 1.

The steady state vector is

$$\vec{v} = \frac{1}{19+13+14}\begin{bmatrix} 19 \\ 13 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 19/46 \\ 13/46 \\ 14/46 \end{bmatrix}$$