Introductory Example

Use an augmented matrix to mimic the elimination method to solve the linear system of equations

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}.$$

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases} \quad -3E_1 + E_2 \to E_2 \quad \begin{cases} 2x_1 - 3x_2 = 8 \\ 10x_2 = -60 \end{cases}$$

$$\tfrac{1}{10}E_2 \to E_2 \quad \begin{cases} 2x_1 - 3x_2 = 8 \\ x_2 = -6 \end{cases}$$

$$3E_2 + E_1 \to E_1 \quad \begin{cases} 2x_1 = -10 \\ x_2 = -6 \end{cases}$$

$$\tfrac{1}{2}E_1 \to E_1 \quad \begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$$

The solution to this system is $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$

Check

$2(-5) - 3(-6) = 8$ ✓

$6(-5) + (-6) = -36$ ✓

Augmented Matrix Gaussian Elimination Method.

$$\begin{array}{c|ccc} & x_1 & x_2 & \text{Constants} \\ \text{1st eq} \to & & & \\ \text{2nd eq} \to & & & \end{array} \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right] \quad -3R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 10 & -60 \end{array}\right]$$

$$\tfrac{1}{10}R_2 \to R_2 \quad \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 1 & -6 \end{array}\right]$$

$$3R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 2 & 0 & -10 \\ 0 & 1 & -6 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -6 \end{array}\right]$$

Implied System

$$\begin{cases} 1x_1 + 0x_2 = -5 \\ 0x_1 + 1x_2 = -6 \end{cases}$$

i.e. $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$

Example

Use the method of Gaussian elimination to find an echelon form of the augmented matrix representation for each of the following systems of equations and use that matrix to determine the solution to the system of equations.

a. $\begin{cases} 2x_1 - 5x_2 - 3x_3 = -23 \\ -5x_1 + x_2 - 2x_3 = -7 \\ x_1 + 3x_2 + x_3 = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ -5 & 1 & -2 & -7 \\ 2 & -5 & -3 & -23 \end{array}\right]$$

$$\begin{array}{l} 5R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & -11 & -5 & -29 \end{array}\right]$$

$$\frac{11}{16}R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & -\frac{47}{16} & -\frac{376}{16} \end{array}\right]$$

New, equivalent, system is:

$$\begin{cases} x_1 + 3x_2 + x_3 = 3 \\ 16x_2 + 3x_3 = 8 \\ -\frac{47}{16}x_3 = \frac{-376}{16} \end{cases}$$

Solve, bottom up, using back-substitution.

$-\frac{47}{16}x_3 = \frac{-376}{16} \Rightarrow x_3 = 8$

$16x_2 + 3(8) = 8 \Rightarrow x_2 = -1$

$x_1 + 3(-1) + (8) = 3 \Rightarrow x_1 = -2$

Check

$2(-2) - 5(-1) - 3(8) = -23$ ✓

$-5(-2) + (-1) - 2(8) = -7$ ✓

$(-2) + 3(-1) + (8) = 3$ ✓

$\therefore$ The solution is $\begin{cases} x_1 = -2 \\ x_2 = -1 \\ x_3 = 8 \end{cases}$.

b.
$$
\begin{aligned}
x_1 - 2x_2 + x_4 &= 4\\
-2x_1 + 3x_2 - 2x_3 + 2x_4 &= -3\\
- x_2 - 2x_3 &= -11\\
5x_1 - 10x_2 - 3x_4 &= -1
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c}
1 & -2 & 0 & 1 & 4\\
-2 & 3 & -2 & 2 & -3\\
0 & -1 & -2 & 0 & -11\\
5 & -10 & 0 & -3 & -1
\end{array}\right]
\begin{array}{c}
2R_1 + R_2 \to R_2\\
-5R_1 + R_4 \to R_4
\end{array}
\left[\begin{array}{cccc|c}
1 & -2 & 0 & 1 & 4\\
0 & -1 & -2 & 4 & 5\\
0 & -1 & -2 & 0 & -11\\
0 & 0 & 0 & -8 & -21
\end{array}\right]
$$

$$
-R_2 + R_3 \to R_3
\left[\begin{array}{cccc|c}
1 & -2 & 0 & 1 & 4\\
0 & -1 & -2 & 4 & 5\\
0 & 0 & 0 & -4 & -16\\
0 & 0 & 0 & -8 & -21
\end{array}\right]
$$

$$
-2R_3 + R_4 \to R_4
\left[\begin{array}{cccc|c}
1 & -2 & 0 & 1 & 4\\
0 & -1 & -2 & 4 & 5\\
0 & 0 & 0 & -4 & -16\\
0 & 0 & 0 & 0 & 11
\end{array}\right]
$$

Mission accomplished!

What does the echelon form of the matrix in part (b) tell you about the solution set to the system of equations modeled by the matrix stated in part (b)?

The implied fourth equation is $0 = 11$. No values for $x_1 - x_4$ are going to make $0 = 11$. So the final system has no solutions which means the original system also has no solutions.

Example

Several augmented row echelon form matrices are given below (and on the next page). For each matrix, identify the pivot columns and state the nature of the solution set for the associated system of equations.

a. $\mathbf{A}=\left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The pivot columns are C_1 and C_3

This system has an unlimited number of solutions.

How do I know? ① Fewer equations than unknowns (2 vs. 3)

② No inconsistencies.

Deeper exploration: 2nd equation:

$3x_3 = 0 \Rightarrow x_3 = 0$

From the first equation we now have: $2x_1 + 5x_2 + 5(0) = -2$

So..., $x_1 = -\frac{5}{2}x_2 - 1$

$\therefore$ The general solution is: $\begin{cases} x_1 = -\frac{5}{2}x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$

x_1 is a dependent variable.
x_2 is a free variable.
x_3 is a fixed variable

Two specific ordered triplets that are solutions are: $(-6, 2, 0)$ and $(19, -8, 0)$

$2(-6) + 5(2) + 5(0) = -2$ ✓ $\quad 2(19) + 5(-8) + 0 = -2$ ✓

$3(0) = 0$ ✓ $\quad 3(0) = 0$ ✓

b. $\mathbf{B}=\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$

Pivot columns: C_1 & C_2

(two unknowns, two pivot columns to the left of the augment line)

$\therefore$ unique solution.

The solution is the order pair $(7, 2)$

you could write: the solution is $\begin{cases} x_1 = 7 \\ x_2 = 2 \end{cases}$

c. $\overset{A}{\mathbf{C}} = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Pivot columns: C_1, C_2, & C_5

Leading entries: $a_{11} = -2$
$a_{22} = 4$
$a_{35} = 6$

Any time the right most column is a pivot column the attendant system of equations has no solutions.

Example

Solve the next three systems using the Gauss-Jordan elimination method. $\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$

$\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 6 & -6 & -1 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & -15 & 5 \end{array}\right]$

$-\frac{1}{15}R_2 \to R_2 \quad \left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & 1 & -1/3 \end{array}\right]$

$-3R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 1 & -1/3 \end{array}\right]$

$\frac{1}{2}R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & -1/3 \end{array}\right]$

$\therefore$ The solution is the ordered pair $\left(-\frac{1}{2}, -\frac{1}{3}\right)$

Check

$2\left(-\frac{1}{2}\right) + 3\left(-\frac{1}{3}\right) = -2$ ✓

$6\left(-\frac{1}{2}\right) - 6\left(-\frac{1}{3}\right) = -1$ ✓

$$\begin{cases} \qquad\quad 2x_2 - 6x_3 = -2 \\ 4x_1 - x_2 + 3x_3 = 1 \\ -x_1 + 3x_2 - 8x_3 = -4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -6 & -2 \\ 4 & -1 & 3 & 1 \\ -1 & 3 & -8 & -4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$ $$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 4 & -1 & 3 & 1 \\ 0 & 2 & -6 & -2 \end{array}\right]$$

$4R_1 + R_2 \to R_2$ $$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 2 & -6 & -2 \end{array}\right]$$

$R_2 \leftrightarrow R_3$ $$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 2 & -6 & -2 \\ 0 & 11 & -29 & -15 \end{array}\right]$$

$-\frac{11}{2}R_2 + R_3 \to R_3$ $$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 2 & -6 & -2 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

$-R_1 \to R_1$
$\frac{1}{2}R_2 \to R_2$
$\frac{1}{4}R_3 \to R_3$ $$\left[\begin{array}{ccc|c} 1 & -3 & 8 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$-8R_3 + R_1 \to R_1$
$3R_3 + R_2 \to R_2$ $$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$3R_2 + R_1 \to R_1$ $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Check

$2(-4) - 6(-1) = -2$ ✓
$4(0) - (-4) + 3(-1) = 1$ ✓
$-(0) + 3(-4) - 8(-1) = -4$ ✓

The solution is $\begin{cases} x_1 = 0 \\ x_2 = -4 \\ x_3 = -1 \end{cases}$

$$\begin{cases} -x_1 + 6x_2 - 2x_3 = 9 \\ 3x_1 - 2x_2 + x_3 + 5x_4 = -1 \\ 2x_1 + 4x_2 - x_3 + 5x_4 = 8 \\ -3x_1 - x_2 + x_3 - 7x_4 = -6 \end{cases}$$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 3 & -2 & 1 & 5 & -1 \\ 2 & 4 & -1 & 5 & 8 \\ -3 & -1 & 1 & -7 & -6 \end{array}\right]$$

$3R_1 + R_2 \to R_2$
$2R_1 + R_3 \to R_3$
$-3R_1 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & -19 & 7 & -7 & -33 \end{array}\right]$$

$-R_2 + R_3 \to R_3$
$\frac{19}{16}R_2 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17/16 & -17/16 & -17/8 \end{array}\right]$$

$\frac{16}{17}R_4 \to R_4$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$2R_3 + R_1 \to R_1$
$5R_3 + R_2 \to R_2$

$$\left[\begin{array}{cccc|c} -1 & 6 & 0 & -2 & 5 \\ 0 & 16 & 0 & 0 & 16 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-1 \cdot R_1 \to R_1$
$\frac{1}{16}R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 1 & -6 & 0 & 2 & -5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$6R_2 + R_1 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The original system is equivalent to: $\begin{cases} x_1 + 2x_4 = 1 \\ x_2 = 1 \\ x_3 - x_4 = -2 \\ 0 = 0 \end{cases}$.

To find the general solution, solve bottom up, and:

- Always solve for the left-most variable
- Back substitute what you've solved for as you move up the list of equations

Eq 3: $x_3 - x_4 = -2 \Rightarrow x_3 = x_4 - 2$

Eq 2: $x_2 = 1$

Eq 1 (which substitutions): $x_1 + 2x_4 = 1 \Rightarrow x_1 = -2x_4 + 1$

Any variable which ends up on the right is a free variable.

So the general solution is: $\begin{cases} x_1 = -2x_4 + 1 \\ x_2 = 1 \\ x_3 = x_4 - 2 \\ x_4 \text{ is free} \end{cases}$

To find specific solutions, let the free variables take on any values you choose and determine the values of the dependent variables.

Letting $x_4 = 0$ we get the solution $(1, 1, -2, 0)$ ✓✓✓✓

Letting $x_4 = 1$ we get the solution $(-1, 1, -1, 1)$ ✓✓✓✓

Example
State the general solution to the following system of equations and rigorously verify the solution. Also, state two specific solutions to the system.

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \quad\;\; x_2 - x_3 + 2x_4 = -3 \end{cases}$$

Eq 2: $x_2 - x_3 + 2x_4 = -3 \Rightarrow x_2 = x_3 - 2x_4 - 3$

Eq 1 (with substitutions): $x_1 - 2(x_3 - 2x_4 - 3) + 2x_3 - 3x_4 = 2$

$\Rightarrow x_1 + x_4 + 6 = 2$

$\Rightarrow x_1 = -x_4 - 4$

$\therefore$ The general solution is $\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$

Letting $x_3 = 0$, $x_4 = 0$ we get the specific solution $(-4, -3, 0, 0)$

Letting $x_3 = 0$, $x_4 = 1$ we get the specific solution $(-5, -5, 0, 1)$

Check $(-4, -3, 0, 0)$

$-4 - 2(-3) + 2(0) - 3(0) = 2$ ✓

$(-3) - 0 + 2(0) = -3$ ✓

Check $(-5, -5, 0, 1)$

$(-5) - 2(-5) + 2(0) - 3(1) = 2$ ✓

$(-5) - (0) + 2(1) = -3$ ✓

Example

State the general solution to the following system of equations and rigorously verify the solution. Also, state two specific solutions to the system.

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \quad\;\; x_2 - x_3 + 2x_4 = -3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right] \; 2R_2 + R_1 \to R_1 \; \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right]$$

The simplified system is:

$$\begin{cases} x_1 + x_4 = -4 \\ x_2 - x_3 + 2x_4 = -3 \end{cases} \text{ which gives us } \begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \end{cases}$$

The general solution is $\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$

A <u>specific solution</u> (letting $x_3 = 2$ and $x_4 = -1$)

is $(-3, 1, 2, -1)$

Check

$(-3) - 2(1) + 2(2) - 3(-1) = 2$ ✓

$(1) - (2) + 2(-1) = -3$ ✓

Correct Solution $x_1 = -3x_2 - 1$

If I get $x_1 = 5x_2 - 1$ replacing x_2 with zero won't catch the error.

Example

State a ***general solution*** to the following system of equations as well as two ***specific solutions*** to the system.

$$\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 5 & -2 \\ -1 & -3 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The equivalent system is $\begin{cases} x_1 + 3x_2 = -1 \\ x_3 = 0 \end{cases}$

General solution: $\begin{cases} x_1 = -3x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$

A specific solution (letting $x_2 = 7$) is $(-22, 7, 0)$.

Check

$2(-22) + 6(7) + 5(0) = -2$ ✓

$-(-22) - 3(7) + 3(0) = 1$ ✓

If you're using specific solutions to check, do not replace the free variable with 0.

Checking General Solution:

$2(-3x_2 - 1) + 6x_2 + 5(0) = -6x_2 - 2 + 6x_2 + 0 = -2$ ✓

$-(-3x_2 - 1) - 3x_2 + 3(0) = 3x_2 + 1 - 3x_2 + 0 = 1$ ✓

Example

Is it possible to find a value of C that makes the following system inconsistent? How do you know?

$$\begin{cases} \quad\quad 2x_2 + x_3 = -2 \\ x_1 - 4x_2 - 4x_3 = C \\ 2x_1 + 12x_2 + 5x_3 = 7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 1 & -4 & -4 & C \\ 2 & 12 & 5 & 7 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 2 & 12 & 5 & 7 \end{array}\right]$$

$$-2R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 20 & 13 & -2C+7 \end{array}\right]$$

$$-10R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & -2C+27 \end{array}\right]$$

We can stop, because we can see that the system will always have a solution. REF is always deep enough to ascertain that. The third equation gives us $x_3 = \dfrac{-2C+27}{3}$ and back-substitution would give us the other two variables

Example

Under what conditions is the following system of equations consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right] \begin{array}{l} -R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & -6 & 9 & 2a+c \end{array}\right]$$

$$3R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & 0 & 0 & -a+3b+c \end{array}\right]$$

Good golly, the last equation is $0 = -a+3b+c$, so the only way this system has a solution is if $-a+3b+c=0$; i.e. $c = a-3b$

Examples

Works

$a=1,\ b=1,\ c=1-3(1)=-2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & -1 & 1 \\ -2 & -8 & 5 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 1 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equivalent to $\begin{cases} x_1 + \frac{7}{2}x_3 = 1 \\ x_2 - \frac{3}{2}x_3 = 0 \\ 0 = 0 \end{cases}$

General solution: $\begin{cases} x_1 = -\frac{7}{2}x_3 + 1 \\ x_2 = \frac{3}{2}x_3 \\ x_3 \text{ is free} \end{cases}$

Given $a=1,\ b=1,\ c=-2$

Doesn't work

$a=1,\ b=1,\ c=7$ not -2

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & -1 & 1 \\ -2 & -8 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Inconsistent: $0 \neq 1$

Application: Balancing Chemical Equations

Ethane and Oxygen combine to produce Carbon Dioxide and steam. Formally, this is represented by the equation $C_2H_6 + O_2 \rightarrow CO_2 + H_2O$. Let's put our newfound skills to use and balance this equation. Speaking of putting things to use … let's use our calculator to find the RREF form of the matrix.

Let's begin with a little vocabulary and concept discussion. Ethane, Carbon Dioxide, etc. are call molecules which in turn are made up of atoms. For example, one Ethane molecule consists of two carbon atoms (C_2) and six hydrogen atoms (H_6). While ethane and oxygen molecules combine two make two new molecules (carbon dioxide and water), the number of each type of atom remains constant. Determining the relative number of each type of molecule required so that the number of atoms is the same before and after the reaction occurs is called balancing the equation.

On the next page I show you type of work I expect to see on your graded homework - this includes details like explicitly defining variables.

Let x_1 be the number of ethane molecules (C_2H_6) in a balanced equation.
Let x_2 be the number of oxygen molecules (O_2) in a balanced equation.
Let x_3 be the number of carbon dioxide molecules (CO_2) in a balanced equation.
Let x_4 be the number of water molecules (H_2O) in a balanced equation.
Our refined equation is now: $x_1 C_2H_6 + x_2 O_2 \rightarrow x_3 CO_2 + x_4 H_2O$

Let's represent each molecules composition by the vector $\begin{bmatrix} \text{\# of carbon atoms} \\ \text{\# of hydrogen atoms} \\ \text{\# of oxygen atoms} \end{bmatrix}$.

Then $x_1 \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

This gives us the system:

$$\begin{cases} 2x_1 - x_3 = 0 \\ 6x_1 - 2x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -7/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{array}\right]$$

So an equivalent system is

$$\begin{cases} x_1 - \frac{1}{3}x_4 = 0 \\ x_2 - \frac{7}{6}x_4 = 0 \\ x_3 - \frac{2}{3}x_4 = 0 \end{cases}$$

which yields the general solution

$$\begin{cases} x_1 = \frac{1}{3}x_4 \\ x_2 = \frac{7}{6}x_4 \\ x_3 = \frac{2}{3}x_4 \\ x_4 \text{ is free} \end{cases}$$

Since we cannot have a fractional number of atoms, we need to choose a positive value for x_4 that results in integers all around. The smallest value that works is $x_4 = 6$ which gives us $x_1 = 2$, $x_2 = 7$, $x_3 = 4$.

$\therefore$ A balanced equation is $2C_2H_6 + 7O_2 \rightarrow 4CO_2 + 6H_2O$

Check:
C: $2(2) = 4(1)$ ✓
H: $2(6) = 6(2)$ ✓
O: $7(2) = 4(2) + 6(1)$ ✓

Application: Network Analysis

A network is most easily thought of as a city street system. The intersections are technically called ***nodes*** or ***junctions*** and each directed stretch of road between intersections is called a ***branch***. Because branches are directed, if there is a two-way street between two intersections the corresponding network will have two branches between the corresponding nodes.

We assign values or variables to each branch; those values and variables could conceptually represent flow-rates or flow-amounts along those branches. In order for the network to be valid, **the total flow into the network must equal the total flow out of the network**.

The values and variables in Figure 1 represent traffic flow rates (vehicles/quarter-hour) in a small section of a city street system. Let's determine the minimum and maximum flow rates through each of the variable branches.

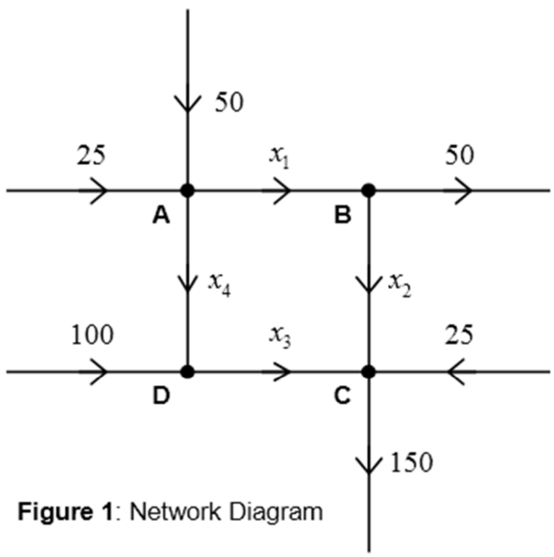

Figure 1: Network Diagram

node	flow rate in	=	flow rate out
A	$25 + 50$	=	$x_1 + x_4$
B	x_1	=	$x_2 + 50$
C	$x_2 + x_3 + 25$	=	150
D	$x_4 + 100$	=	x_3

We can model the network with the following system of equations.

$$\begin{cases} x_1 + x_4 = 75 \\ x_1 - x_2 = 50 \\ x_2 + x_3 = 125 \\ x_3 - x_4 = 100 \end{cases}.$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 1 & -1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 125 \\ 0 & 0 & 1 & -1 & 100 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the general solution is

$$\begin{cases} x_1 = -x_4 + 75 \\ x_2 = -x_4 + 25 \\ x_3 = x_4 + 100 \\ x_4 \text{ is free} \end{cases}.$$

Since none of the flow rates can be negative, x_4 must be at least 0 and no more than 25 ($x_4 > 25 \Rightarrow x_2 < 0$). This gives us:

$$\begin{cases} 50 \le x_1 \le 75 \\ 0 \le x_2 \le 25 \\ 100 \le x_3 \le 125 \\ 0 \le x_4 \le 25 \end{cases}.$$