CHECK $(-5, -6)$

Introductory Example

Use an augmented matrix to mimic the elimination method to solve the linear system of equations $\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}$.

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases} \qquad -3E_1 + E_2 \to E_2 \qquad \begin{cases} 2x_1 - 3x_2 = 8 \\ 10x_2 = -60 \end{cases}$$

Augmented Matrix

Augment line

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right] \qquad -3R_1 + R_2 \to R_2 \qquad \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 10 & -60 \end{array}\right]$$

MAIN DIAGONAL

In (ROW) ECHELON FORM, EVERY ENTRY BELOW THE MAIN DIAGONAL IS ZERO.

The goal in Gaussian Elimination is ECHELON FORM

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 10x_2 = -60 \end{cases}$$

From Eq 2, $x_2 = -6$

Backsub to Eq 1.

$$2x_1 - 3(-6) = 8$$

$$x_1 = -5$$

The solution is $(-5, -6)$.

Check $(-2, -1, 8)$

Example

Use the method of Gaussian elimination to find an echelon form of the augmented matrix representation for each of the following systems of equations and use that matrix to determine the solution to the system of equations.

a. $\begin{cases} 2x_1 - 5x_2 - 3x_3 = -23 \\ -5x_1 + x_2 - 2x_3 = -7 \\ x_1 + 3x_2 + x_3 = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ -5 & 1 & -2 & -7 \\ 2 & -5 & -3 & -23 \end{array}\right]$$

$$\begin{array}{l} 5R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & -11 & -5 & -29 \end{array}\right]$$

$$\begin{array}{l} 11R_2 \to R_2 \\ 16R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 176 & 33 & 88 \\ 0 & -176 & -80 & -464 \end{array}\right]$$

$$R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 176 & 33 & 88 \\ 0 & 0 & -47 & -376 \end{array}\right]$$

$$\begin{cases} x_1 + 3x_2 + x_3 = 3 \\ 176x_2 + 33x_3 = 88 \\ -47x_3 = -376 \end{cases}$$

From Eq 3: $x_3 = 8$

Subbing into Eq 2:

$$176x_2 + 33(8) = 88$$
$$x_2 = -1$$

Subbing into Eq 1

$$x_1 + 3(-1) + 8 = 3$$
$$x_1 = -2$$

The solution is $(-2, -1, 8)$

The solution set is $\{(-2, -1, 8)\}$

b. $\begin{cases} x_1 - 2x_2 + x_4 = 4 \\ -2x_1 + 3x_2 - 2x_3 + 2x_4 = -3 \\ -x_2 - 2x_3 = -11 \\ 5x_1 - 10x_2 - 3x_4 = -1 \end{cases}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ -2 & 3 & -2 & 2 & -3 \\ 0 & -1 & -2 & 0 & -11 \\ 5 & -10 & 0 & -3 & -1 \end{array}\right]$$

$2R_1 + R_2 \to R_2$
$-5R_1 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$$

$-1R_2 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$$

Let's manifest the inconsistency

$-2R_3 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

no way, no how does $0 = 11$

What does the echelon form of the matrix in part (b) tell you about the solution set to the system of equations modeled by the matrix stated in part (b)?

The last two rows are contradictory $\begin{cases} -4x_4 = -16 \\ -8x_4 = -21 \end{cases}$

x_4 cannot be both 4 and 21/8.

Since this last system has no solution, the equivalent original system also has no solution; i.e. the system is inconsistent.

Pivot entries circled

Example

Several augmented row echelon form matrices are given below (and on the next page). For each matrix, identify the pivot columns and state the nature of the solution set for the associated system of equations.

a. $\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The pivot columns are C1 and C3

Since there are fewer pivot columns than variables (2 < 3) there are either no solutions or an infinite number of solutions. Since no pivot entry is to the right of the augment line there are no inconsistencies

Ergo, there are an unlimited number of solutions.

b. $\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$

The pivot columns are C1 and C2.

Since there are an equal number of pivot columns and variables the System has exactly one Solution.

c. $\mathbf{C} = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The pivot columns are C_1, C_2 and C_5. Since the last column is a pivot column, this system (and any equivalent system) has no solutions.

Example

Check $(-1/2, -1/3)$

Solve the next three systems using the Gauss-Jordan elimination method. $\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$

$\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 6 & -6 & -1 \end{array}\right]$ $-3R_1 + R_2 \to R_2$ $\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & -15 & 5 \end{array}\right]$

$\frac{3}{15} R_2 + R_1 \to R_1$ $\left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & -15 & 5 \end{array}\right]$

$\frac{1}{2} R_1 \to R_1$

$-\frac{1}{15} R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & -1/3 \end{array}\right]$

$\begin{cases} x_1 = -1/2 \\ x_2 = -1/3 \end{cases}$

The solution is $(-1/2, -1/3)$

Check $(0, -4, -1)$

$$\begin{cases} \quad\quad\; 2x_2 - 6x_3 = -2 \\ 4x_1 - x_2 + 3x_3 = 1 \\ -x_1 + 3x_2 - 8x_3 = -4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -6 & -2 \\ 4 & -1 & 3 & 1 \\ -1 & 3 & -8 & -4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$
$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 4 & -1 & 3 & 1 \\ 0 & 2 & -6 & -2 \end{array}\right]$$

$4R_1 + R_2 \rightarrow R_2$
$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 2 & -6 & -2 \end{array}\right]$$

$-\frac{11}{2}R_3 \rightarrow R_3$
$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & -11 & 33 & 11 \end{array}\right]$$

$1 \cdot R_2 + R_3 \rightarrow R_3$
$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

$\frac{1}{4}R_3 \rightarrow R_3$
$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$8R_3 + R_1 \rightarrow R_1$
$29R_3 + R_2 \rightarrow R_2$
$$\left[\begin{array}{ccc|c} -1 & 3 & 0 & -12 \\ 0 & 11 & 0 & -44 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$-1R_1 \rightarrow R_1$
$\frac{1}{11}R_2 \rightarrow R_2$
$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$3R_2 + R_1 \rightarrow R_1$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $(0, -4, -1)$

$$\begin{cases} -x_1 + 6x_2 - 2x_3 = 3 \\ 3x_1 - 2x_2 + x_3 + 5x_4 = -2 \\ 2x_1 + 4x_2 - x_3 + 5x_4 = 8 \\ -3x_1 - x_2 + x_3 - 7x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 3 & -2 & 1 & 5 & -2 \\ 2 & 4 & -1 & 5 & 8 \\ -3 & -1 & 1 & -7 & 0 \end{array}\right) \begin{array}{l} 3R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \\ -3R_1 + R_4 \to R_4 \end{array} \left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 0 & 16 & -5 & 5 & 7 \\ 0 & 16 & -5 & 5 & 14 \\ 0 & -19 & 7 & -7 & -9 \end{array}\right)$$ Inconsistency

The system is inconsistent – i.e. it has no solutions.

$16x_2 - 5x_3 + 5x_4$ cannot simultaneously sum to 7 and 14.

Two Specific Solutions

$x_3 = 0$, $x_4 = 0$ $\Rightarrow$ $x_1 = -4$, $x_2 = -3$

$(-4, -3, 0, 0)$

$x_3 = 1$, $x_4 = 0$ $\Rightarrow$ $x_1 = -4$, $x_2 = -2$

$(-4, -2, 1, 0)$

Example

State the general solution to the following system of equations and rigorously verify the solution. Also, state two specific solutions to the system.

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \quad\;\; x_2 - x_3 + 2x_4 = -3 \end{cases}$$

This system has fewer equations than unknowns, so it has either no solutions or unlimited solutions.

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right) \xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right)$$

A equivalent system to the original system is

$$\begin{cases} x_1 \qquad\qquad + x_4 = -4 \\ \quad x_2 - x_3 + 2x_4 = -3 \end{cases}$$

Solve bottom up; always solve for the left most variable (and substitute along the way)

From Eq 2 $\quad x_2 = x_3 - 2x_4 - 3$

From Eq 1 $\quad x_1 = -x_4 - 4$

The solution is
$$\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

The way we stated are solution, x_3 & x_4 are free variables and x_1 & x_2 are dependent variables (basic variables)

Check

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ x_2 - x_3 + 2x_4 = -3 \end{cases}$$

$$\begin{cases} (-x_4 - 4) - 2(x_3 - 2x_4 - 3) + 2x_3 - 3x_4 \stackrel{?}{=} 2 \\ (x_3 - 2x_4 - 3) - x_3 + 2x_4 \stackrel{?}{=} -3 \end{cases}$$

$$\begin{cases} 2 = 2 \;\checkmark \\ -3 = -3 \;\checkmark \end{cases}$$

Example

State a ***general solution*** to the following system of equations as well as two ***specific solutions*** to the system.

$$\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 5 & -2 \\ -1 & -3 & 3 & 1 \end{array}\right] \xrightarrow[\text{RREF}]{\text{Calc}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The equivalent system is $\begin{cases} x_1 + 3x_2 = -1 \\ x_3 = 0 \end{cases}$

x_3 is a fixed variable

$x_1 = -3x_2 - 1$; x_1 is a dependent (basic) variable

x_2 is a free variable

The solution is $\begin{cases} x_1 = -3x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$

Two specific solutions are

$(-1, 0, 0)$ and $(-4, 1, 0)$

choice

Example

Is it possible to find a value of C that makes the following system inconsistent? How do you know?

$$\begin{cases} \quad\;\; 2x_2 + x_3 = -2 \\ x_1 - 4x_2 - 4x_3 = C \\ 2x_1 + 12x_2 + 5x_3 = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 1 & -4 & -4 & C \\ 2 & 12 & 5 & 7 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 2 & 12 & 5 & 7 \end{array}\right)$$

$$-2R_1 + R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 20 & 13 & -2C+7 \end{array}\right)$$

$$-10R_2 + R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & -2C+27 \end{array}\right)$$

This system is never inconsistent.

There is no "0 = not zero" row.

Example

Under what conditions is the following system of equations consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right) \begin{array}{l} -R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & -6 & 9 & 2a+c \end{array}\right)$$

$$3R_2 + R_3 \to R_3 \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & 0 & 0 & -a+3b+c \end{array}\right)$$

The system is consistent iff $-a + 3b + c = 0$

Show on your calculator

Consistent

$a = 4$

$b = 2$

$c = -2$

$-4 + 3(2) + (-2) = 0$

inconsistent

$a = 4$

$b = 2$

$c =$ any other number but -2.

Application: Balancing Chemical Equations

Ethane and Oxygen combine to produce Carbon Dioxide and steam. Formally, this is represented by the equation $C_2H_6 + O_2 \to CO_2 + H_2O$. Let's put our newfound skills to use and balance this equation. Speaking of putting things to use … let's use our calculator to find the RREF form of the matrix.

Let x_1 be the number of C_2H_6 molecules used in the mixture
Let x_2 be the number of O_2 molecules used in the mixture
Let x_3 be the resultant number of CO_2 molecules
Let x_4 be the resultant number of H_2O molecules

Balance carbon: $2x_1 = x_3$
Balance hydrogen: $6x_1 = 2x_4$
Balance oxygen: $2x_2 = 2x_3 + x_4$

$$\Rightarrow \begin{cases} 2x_1 - x_3 = 0 \\ 6x_1 - 2x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -7/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{array}\right]$$

General Solution:
$$\begin{cases} x_1 = \frac{1}{3}x_4 \\ x_2 = \frac{7}{6}x_4 \\ x_3 = \frac{2}{3}x_4 \\ x_4 \text{ is free} \end{cases}$$

You can't have fractional atoms (without calamitous results)

So let $x_4 = 6$

A balanced equation is $2C_2H_6 + 7O_2 \to 4CO_2 + 6H_2O$

Application: Network Analysis

A network is most easily thought of as a city street system. The intersections are technically called ***nodes*** or ***junctions*** and each directed stretch of road between intersections is called a ***branch***. Because branches are directed, if there is a two-way street between two intersections the corresponding network will have two branches between the corresponding nodes.

We assign values or variables to each branch; those values and variables could conceptually represent flow-rates or flow-amounts along those branches. In order for the network to be valid, **the total flow into the network must equal the total flow out of the network**.

The values and variables in Figure 1 represent traffic flow rates (vehicles/quarter-hour) in a small section of a city street system. Let's determine the minimum and maximum flow rates through each of the variable branches.

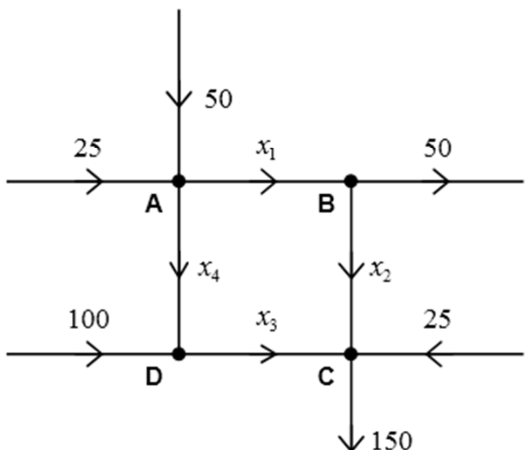

Figure 1: Network Diagram

nodal Equations

CARS IN = CARS OUT

A: $75 = x_1 + x_4$

B: $x_1 = x_2 + 50$

C: $x_2 + 25 + x_3 = 150$

D: $x_4 + 100 = x_3$

System:

$$\begin{cases} x_1 + x_4 = 75 \\ x_1 - x_2 = 50 \\ x_2 + x_3 = 125 \\ x_3 - x_4 = 100 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 1 & -1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 125 \\ 0 & 0 & 1 & -1 & 100 \end{array}\right] \xrightarrow[\text{RREF}]{\text{Calculator}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + x_4 = 75 \\ x_2 + x_4 = 25 \\ x_3 - x_4 = 100 \end{cases}$$

The general solution is

$$\begin{cases} x_1 = -x_4 + 75 \\ x_2 = -x_4 + 25 \\ x_3 = x_4 + 100 \\ x_4 \text{ is free} \end{cases}$$

We can't have a negative car flow rate.

So $0 \le x_4 \le 25$

∴ The flow rate constraints are

$$\begin{cases} 50 \le x_1 \le 75 \\ 0 \le x_2 \le 25 \\ 100 \le x_3 \le 125 \\ 0 \le x_4 \le 25 \end{cases}$$