Introductory Example

Use an augmented matrix to mimic the elimination method to solve the linear system of equations $\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}$.

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases} \Longrightarrow \begin{cases} 2x_1 - 3x_2 = 8 \\ 3(6x_1 + x_2) = 3(-36) \end{cases}$$

$$\Longrightarrow \begin{cases} 2x_1 - 3x_2 = 8 \\ 18x_1 + 3x_2 = -108 \end{cases}$$

$$20x_1 = -100$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right] \xrightarrow{3R_2 \to R_2} \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 18 & 3 & -108 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 20 & 0 & -100 \end{array}\right]$$

$$20x_1 = -100$$

Gauss-Jordan want the zero here.

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \underbrace{\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 10 & -60 \end{array}\right]}_{\text{(row) echelon form}}$$

$$10x_2 = -60 \Longrightarrow x_2 = -6$$

$$2x_1 - 3(-6) = 8 \Longrightarrow x_1 = -5$$

The solution is $(-5, -6)$.

main diagonal

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right]$$

we always want zeros below the main diagonal

MTH 261 – Mr. Simonds' class

zeros here first

zero here second

Example

Use the method of Gaussian elimination to find an echelon form of the augmented matrix representation for each of the following systems of equations.

a. $\begin{cases} 2x_1 - 5x_2 - 3x_3 = -23 \\ -5x_1 + x_2 - 2x_3 = -7 \\ x_1 + 3x_2 + x_3 = 3 \end{cases}$ $(-2, -1, 8)$

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ -5 & 1 & -2 & -7 \\ 2 & -5 & -3 & -23 \end{array}\right]$$

$$\begin{array}{l} 5R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & -11 & -5 & -29 \end{array}\right]$$

$16x = 11$

$x = \frac{11}{16}$

$$\frac{11}{16}R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & -\frac{47}{16} & -\frac{47}{2} \end{array}\right]$$

$$-\frac{47}{16}x_3 = -\frac{47}{2} \Rightarrow x_3 = 8$$

$$16x_2 + 3(8) = 8 \Rightarrow x_2 = -1$$

$$x_1 + 3(-1) + (8) = 3 \Rightarrow x_1 = -2$$

The solution is $(-2, -1, 8)$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \cancel{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{aligned} x_1 - 2x_2 + x_4 &= 4 \\ -2x_1 + 3x_2 - 2x_3 + 2x_4 &= -3 \\ - x_2 - 2x_3 &= -11 \\ 5x_1 - 10x_2 - 3x_4 &= -1 \end{aligned}$$

$$\begin{array}{ccccc} \text{PC} & \text{PC} & & \text{PC} & \text{PC} \\ \downarrow & \downarrow & & \downarrow & \downarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ -2 & 3 & -2 & 2 & -3 \\ 0 & -1 & -2 & 0 & -11 \\ 5 & -10 & 0 & -3 & -1 \end{array}\right] \begin{matrix} 2R_1 + R_2 \to R_2 \\ -5R_1 + R_4 \to R_4 \end{matrix} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$$

$$-R_2 + R_3 \to R_3 \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$$

Rogue leading entry, not in echelon form

$$-2R_3 + R_4 \to R_4 \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

contradiction/inconsistency $\longrightarrow$ $0 = 11$ I don't think so $\longleftarrow$ that row!

What does the echelon form of the matrix in part (b) tell you about the solution set to the system of equations modeled by the matrix stated in part (b)?

The system of equations has no solutions. No values for $x_1 - x_4$ are going to make $0 = 11$!

Mr. Jordan also insists on zeros above the leading entries. Make these right-to-left

Example

Use the echelon form matrix found in part (a) of the last example to find the solution to the system.

Been there, done that

and Mr. Jordan insists that leading entries all be 1s.

from page 7

$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 4 & -4 \end{array}\right] \xrightarrow{\frac{1}{4}R_3 \to R_3} \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{l} 8R_3 + R_1 \to R_1 \\ 3R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} -1 & 3 & 0 & -12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$-3R_2 + R_1 \to R_1 \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$-R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $(0, -4, -1)$

Example

Several augmented row echelon form matrices are given below (and on the next page). For each matrix, identify the pivot columns and state the nature of the solution set for the associated system of equations.

a. $\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Pivot columns contain leading entries (eventually) when the matrix is in REF.

a. columns I & III

Three variables

Two pivot column to the left of the augment line and no pivot columns to the right of the augment line

no inconsistencies, fewer pivot columns than variables

∴ unlimited number of solutions

The system is consistent, the equations are dependent.

b. $\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$

Pivot columns: 1 & 2

Two variables

Two pivot columns to the left of the augment line

equal number

There is a unique solution to the system.

The system is consistent, the equations are independent

c. $\mathbf{C} = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Pivot Columns: 1, 2, and 5.

Whenever a pivot column occurs to the right of an augment line, the attendant system of equations is inconsistent (no solutions)

$0 \neq 6$

Example

Solve the next three systems using the Gauss-Jordan elimination method. $\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 6 & -6 & -1 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & -15 & 5 \end{array}\right]$$

$$-15x_2 = 5 \Rightarrow x_2 = -\frac{1}{3}$$

$$2x_1 + 3\left(-\frac{1}{3}\right) = -2 \Rightarrow x_1 = -\frac{1}{2}$$

The solution is $\left(-\frac{1}{2}, -\frac{1}{3}\right)$

$$\begin{cases} \quad\;\; 2x_2 - 6x_3 = -2 \\ 4x_1 - x_2 + 3x_3 = 1 \\ -x_1 + 3x_2 - 8x_3 = -4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -6 & -2 \\ 4 & -1 & 3 & 1 \\ -1 & 3 & -8 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 4 & -1 & 3 & 1 \\ 0 & 2 & -6 & -2 \end{array}\right]$$

$$\begin{array}{c} 4R_1 + R_2 \to R_2 \\ \tfrac{1}{2}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 11 & -29 & -15 \end{array}\right]$$

$$-11R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

$$4x_3 = -4 \implies x_3 = -1$$

$$x_2 - 3(-1) = -1 \implies x_2 = -4$$

$$-x_1 + 3(-4) - 8(-1) = -4 \implies x_1 = 0$$

The solution is $(0, -4, -1)$.

Stopping at REF is the Gaussian method. Let's go back to page 4 and finish the Jordan part

$$\begin{cases} -x_1 + 6x_2 - 2x_3 = 3 \\ 3x_1 - 2x_2 + x_3 + 5x_4 = -2 \\ 2x_1 + 4x_2 - x_3 + 5x_4 = 8 \\ -3x_1 - x_2 + x_3 - 7x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 3 & -2 & 1 & 5 & -2 \\ 2 & 4 & -1 & 5 & 8 \\ -3 & -1 & 1 & -7 & 0 \end{array}\right] \begin{array}{c} 3R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \\ -3R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 0 & 16 & -5 & 5 & 7 \\ 0 & 16 & -5 & 5 & 14 \\ 0 & -19 & 7 & -7 & -9 \end{array}\right]$$

$$-R_2 + R_3 \to R_3 \left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 0 & 16 & -5 & 5 & 7 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & -19 & 7 & -7 & -9 \end{array}\right]$$

inconsistency: $0 \neq 7$

The system has no solutions

Example

State the general solution to the following system of equations and rigorously verify the solution. Also, state two specific solutions to the system.

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \qquad x_2 - x_3 + 2x_4 = -3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right]$$

$$\begin{cases} x_1 + x_4 = -4 \\ x_2 - x_3 + 2x_4 = -3 \end{cases} \Rightarrow \underbrace{\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \end{cases}}$$

The solution set; x_3 and x_4 are free to assume any values.

verification

Original Eq 1: $(-x_4 - 4) - 2(x_3 - 2x_4 - 3) + 2x_3 - 3x_4 \stackrel{?}{=} 2$

$$-\cancel{x_4} - 4 - \cancel{2x_3} + \cancel{4x_4} + 6 + \cancel{2x_3} - \cancel{3x_4} \stackrel{?}{=} 2$$

$$2 = 2 \checkmark$$

Original Eq 2: $(\cancel{x_3} - \cancel{2x_4} - 3) - \cancel{x_3} + \cancel{2x_4} \stackrel{?}{=} -3$

$$-3 = -3 \checkmark$$

We generate specific solutions by replacing the free variable with specific values

$x_3 = -2$

$x_4 = 1$

$x_1 = -(1) - 4 = -5$

$x_2 = -2 - 2(1) - 3 = -7$

$x_3 = 0$

$x_4 = 0$

$x_1 = -(0) - 4 = -4$

$x_2 = 0 - 2(0) - 3 = -3$

Two specific solutions are $(-5, -7, -2, 1)$ and $(-4, -3, 0, 0)$

Example
State a ***general solution*** to the following system of equations as well as two ***specific solutions*** to the system.

$$\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 5 & -2 \\ -1 & -3 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 3x_2 = -1 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_2 - 1 \\ x_2 = \text{free} \\ x_3 = 0 \end{cases}$$

general solution

Check

$$2(-3x_2 - 1) + 6x_2 + 5(0) \stackrel{?}{=} -2$$

$$-6x_2 - 2 + 6x_2 = -2$$

$$-(-3x_2 - 1) - 3x_2 + 3(0) = 1$$

$$3x_2 + 1 - 3x_2 = 1$$

$x_2 = 0$

$x_1 = -3(0) - 1$
$= -1$

$x_2 = 1$

$x_1 = -3(1) - 1$
$= -4$

Two specific solutions are $(-1, 0, 0)$ and $(-4, 1, 0)$

Example

Is it possible to find a value of C that makes the following system inconsistent? How do you know?

$$\begin{cases} \quad\;\; 2x_2 + x_3 = -2 \\ x_1 - 4x_2 - 4x_3 = C \\ 2x_1 + 12x_2 + 5x_3 = 7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 1 & -4 & -4 & C \\ 2 & 12 & 5 & 7 \end{array}\right] \; R_1 \leftrightarrow R_2 \; \left[\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 2 & 12 & 5 & 7 \end{array}\right]$$

$$-2R_1 + R_3 \to R_3 \; \left[\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 20 & 13 & -2C+7 \end{array}\right]$$

$$-10R_2 + R_3 \to R_3 \; \left[\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 0 & & -2C+27 \end{array}\right]$$

The matrix has three pivot columns <u>to</u> <u>the</u> <u>left</u> of the augment line. Since the system has three variables, the system will have a unique solution point regardless of the value of C. It is impossible for this system to be inconsistent.

Example

Under what conditions is the following system of equations consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right] \begin{array}{l} -R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & -6 & 9 & 2a+c \end{array}\right]$$

$$3R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & 0 & 0 & 3(-a+b)+2a+c \end{array}\right]$$

$\therefore$ The system will be inconsistent iff

$$3(-a+b) + 2a + c \neq 0$$

$\therefore$ The system will be inconsistent iff

$$c \neq a - 3b$$

mini-check

$a = 7$, $b = 12$, $c \neq 7 - 36 = -29$

$c = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 1 & 3 & -1 & 12 \\ -2 & -8 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Leftarrow \text{inconsistency}$$

AS opposed to

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 1 & 3 & -1 & 12 \\ -2 & -8 & 5 & -29 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 9/2 \\ 0 & 1 & -3/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$