Since the solution to the last system $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$ is $(-5,-6)$, $(-5,-6)$ is also the solution to the original system.

Introductory Example

Use an augmented matrix to mimic the elimination method to solve the linear system of equations $\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}$.

$$\left(\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right) \xrightarrow{-3R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 10 & -60 \end{array}\right)$$

$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \left(\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 1 & -6 \end{array}\right) \text{ (Row) Echelon Matrix}$$

$$\xrightarrow{3R_2 + R_1 \to R_1} \left(\begin{array}{cc|c} 2 & 0 & -10 \\ 0 & 1 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -6 \end{array}\right) \text{ Reduced Echelon Form}$$

Each of these Equivalent Matrices represent Equivalent Systems; i.e. Systems have the same solution set.

The Gaussian elimination process for solving systems of linear equations is predicated upon the three ***elementary row operations***. The three elementary row operations are:

1. Interchanging two rows of the matrix
2. Replacing one row of the matrix with a non-zero multiple of itself
3. Replacing one row of a matrix with itself added to a multiple of another row of the matrix

Two matrices are said to be ***row equivalent*** if one matrix can be transformed into the other via a series of elementary row operations. The first non-zero entry of any row of a matrix is called ***the leading entry*** of that row. A matrix is said to be in ***(row) echelon form*** if the matrix satisfies both of the following properties.

i. Every row that contains nothing but zeros occurs at the bottom of the matrix.

ii. The leading entry of any non-zero row appears to the right of the leading entry in the row directly above it.

The process of transforming a matrix into a row equivalent matrix of echelon form is called the ***Gaussian elimination process***.

Example
Use the method of Gaussian elimination to find an echelon form of the augmented matrix representation for each of the following systems of equations.

a. $\begin{cases} 2x_1 - 5x_2 - 3x_3 = -23 \\ -5x_1 + x_2 - 2x_3 = -7 \\ x_1 + 3x_2 + x_3 = 3 \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ -5 & 1 & -2 & -7 \\ 2 & -5 & -3 & -23 \end{array}\right)$$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{5R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & -11 & -5 & -29 \end{array}\right)$$

$$\xrightarrow{\frac{11}{16}R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & -\frac{47}{16} & -\frac{376}{16} \end{array}\right)$$ Technically Complete

$$\xrightarrow{-\frac{16}{47}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & 1 & 8 \end{array}\right)$$

b.
$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & & & + & x_4 & = & 4 \\ -2x_1 & + & 3x_2 & - & 2x_3 & + & 2x_4 & = & -3 \\ & - & x_2 & - & 2x_3 & & & = & -11 \\ 5x_1 & - & 10x_2 & & & - & 3x_4 & = & -1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ -2 & 3 & -2 & 2 & -3 \\ 0 & -1 & -2 & 0 & -11 \\ 5 & -10 & 0 & -3 & -1 \end{array}\right) \xrightarrow[-5R_1+R_4\to R_4]{2R_1+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right)$$

pivot entries

$$\xrightarrow{-1\cdot R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{4}R_3\to R_3} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right)$$

$$\xrightarrow{8R_3+R_4\to R_4} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right)$$

What does the echelon form of the matrix in part (b) tell you about the solution set to the system of equations modeled by the matrix stated in part (b)?

The 4^{th} Equation in the reduced system is $0 = 11$, Ain't no values of x_1, x_2, x_3, x_4 gonna make $0 = 11$. Since the reduced system has no solutions, the equivalent original system also has no solutions.

Example

Use the echelon form matrix found in part (a) of the last example to find the solution to the system.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & 1 & 8 \end{array}\right) \Longrightarrow \begin{cases} x_1 + 3x_2 + x_3 = 3 \\ 16x_2 + 3x_3 = 8 \\ x_3 = 8 \end{cases}$$

Back sub bottom-to-top

$$x_3 = 8 \Longrightarrow 16x_2 + 3(8) = 8$$

$$\Longrightarrow x_2 = -1$$

$$x_2 = -1, x_3 = 8 \Longrightarrow x_1 + 3(-1) + 8 = 3$$

$$\Longrightarrow x_1 = -2$$

The solution to the original system is $(-2, -1, 8)$

Check all three original equations.

Definitions and Fundamental Facts

A ***pivot position*** in a matrix is an entry position that corresponds to a leading entry in a row echelon form of the matrix. Any column that contains a pivot entry is called a ***pivot column***.

If A is an augmented matrix representing a system of equation with n variables, then:

- the system of equations has exactly one solution if and only if A has exactly n pivot columns all of which lie to the left of the augment line.
- the system of equations has no solutions if and only if the right-most column of A is a pivot column.
- the system of equations has an unlimited number of solutions if and only if there are less than n pivot columns in A and the right-most column of A is not a pivot column. .

A system of equations with at least one solution is called ***consistent***. A system of equations with no solutions is called ***inconsistent***.

The solution set for (a)

is: $\begin{cases} x_1 = -\frac{5}{2}x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$

Example

Several augmented row echelon form matrices are given below. For each matrix, identify the pivot columns and state the nature of the solution set for the associated system of equations.

a. $\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Pivot columns: C1, C3

1) Fewer pivot columns than variables

2) no inconsistencies.

$\therefore$ unlimited # of solutions

Eq. 2: $3x_3 = 0 \Rightarrow x_3 = 0$

Eq 1: $x_3 = 0 \Rightarrow 2x_1 + 5x_2 = -2 \Rightarrow x_1 = \frac{-5}{2}x_2 - 1$

b. $\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$

Pivot columns: C1, C2

1) Same # of pivot columns as variables.

2) all pivot columns are to the left of the augment line

$\therefore$ Exactly one solution

c. $\mathbf{C} = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Pivot columns: C1, C2, C5

Eq. 3 is a contradiction $0 \neq 6$

$\therefore$ no solutions

Gauss-Jordan Elimination Method

The ***Gauss-Jordan elimination method*** follows the same row manipulation rules as Gaussian elimination. What distinguishes the two methods is that the Gaussian method stops once you've reduced the matrix to row echelon form where-as the Gauss-Jordanian method requires you to further reduce the matrix to ***reduced (row) echelon form***. That is, the simplified matrix must meet each of the following properties:

i. Every row that contains nothing but zeros occurs at the bottom of the matrix.

ii. The pivot entry of any non-zero row appears to the right of the pivot entry in the row directly above it.

iii. Other than the actual pivot entry, every entry in a pivot column is 0.

iv. Every pivot entry is 1.

When manipulating a matrix into reduced row echelon form, there are essentially three tasks that need to be completed. In the list below, you always want to complete task A first. Tasks B and C can be done in either order. The tasks refer to the ***main diagonal***; this is the left-to-right diagonal that starts in the upper left-hand corner of the matrix.

A. Manipulate every entry below the main diagonal to zero. This task should be worked left-to-right; i.e., first create the necessary zeros in the first column, then the second column, then the third column, etc.

B. Manipulate the entries above the pivot entries to zero. This task should be worked right-to-left.

C. Multiply each of the rows by the necessary constants so that each pivot entry is 1.

Example

Solve the next three systems using the Gauss-Jordan elimination method.

$$\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$$

MAIN DIAGONAL

$$\left(\begin{array}{cc|c} 2 & 3 & -2 \\ 6 & -6 & -1 \end{array}\right) \xrightarrow{-3R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & -15 & 5 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2+R_1 \to R_1} \left(\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & -15 & 5 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{15}R_2 \to R_2]{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & -1/3 \end{array}\right)$$

$$\begin{cases} \quad\quad\; 2x_2 - 6x_3 = -2 \\ 4x_1 - x_2 + 3x_3 = 1 \\ -x_1 + 3x_2 - 8x_3 = -4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & -6 & -2 \\ 4 & -1 & 3 & 1 \\ -1 & 3 & -8 & -4 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 4 & -1 & 3 & 1 \\ 0 & 2 & -6 & -2 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}R_3 \to R_3]{4R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 1 & -3 & -1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 11 & -29 & -15 \end{array}\right)$$

$$\xrightarrow{-11R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 4 & -4 \end{array}\right)$$

$$\begin{array}{l} 2R_3 + R_1 \to R_1 \\ \frac{3}{4}R_3 + R_2 \to R_2 \end{array} \left(\begin{array}{ccc|c} -1 & 3 & 0 & -12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 4 & -4 \end{array}\right)$$

$$-3R_2 + R_1 \to R_1 \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 4 & -4 \end{array}\right)$$

$$\begin{array}{l} -1 \cdot R_1 \to R_1 \\ \frac{1}{4}R_3 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

The solution is $(0, -4, -1)$

$$\begin{cases} -x_1 + 6x_2 - 2x_3 = 3 \\ 3x_1 - 2x_2 + x_3 + 5x_4 = -2 \\ 2x_1 + 4x_2 - x_3 + 5x_4 = 8 \\ -3x_1 - x_2 + x_3 - 7x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 3 & -2 & 1 & 5 & -2 \\ 2 & 4 & -1 & 5 & 8 \\ -3 & -1 & 1 & -7 & 0 \end{array}\right) \begin{array}{l} 3R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \\ -3R_1+R_4 \to R_4 \end{array} \left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 0 & 16 & -5 & 5 & 7 \\ 0 & 16 & -5 & 5 & 14 \\ 0 & -19 & 7 & -7 & -9 \end{array}\right)$$

$$\begin{array}{l} -R_2+R_3 \to R_3 \\ \frac{19}{16}R_2+R_4 \to R_4 \end{array} \left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 0 & 16 & -5 & 5 & 7 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & \frac{17}{16} & -\frac{17}{16} & -\frac{11}{16} \end{array}\right)$$

$$R_3 \leftrightarrow R_4 \left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 3 \\ 0 & 16 & -5 & 5 & 7 \\ 0 & 0 & \frac{17}{16} & -\frac{17}{16} & -\frac{11}{16} \\ 0 & 0 & 0 & 0 & 7 \end{array}\right)$$

$$\begin{array}{l} -\frac{3}{7}R_4+R_1 \to R_1 \\ -R_4+R_2 \to R_2 \\ -\frac{11}{112}R_4+R_3 \to R_3 \end{array} \left(\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 0 \\ 0 & 16 & -5 & 5 & 0 \\ 0 & 0 & \frac{17}{16} & -\frac{17}{16} & 0 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right)$$

$$\begin{array}{l} \frac{32}{17}R_3+R_1 \to R_1 \\ \frac{80}{17}R_3+R_2 \to R_2 \end{array} \left(\begin{array}{cccc|c} -1 & 6 & 0 & -2 & 0 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & \frac{17}{16} & -\frac{17}{16} & 0 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right)$$

$$\begin{array}{l} -\frac{6}{16}R_2+R_1 \to R_1 \\ \frac{16}{17}R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 0 & -2 & 0 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right)$$

$$-R_1 \to R_1$$
$$\frac{1}{16}R_2 \to R_2, \quad \frac{1}{7}R_4 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{bmatrix} -x_4-4 \\ x_3-2x_4-3 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \\ 0 \\ 0 \end{bmatrix}$$

Definitions and Fundamental Facts

Suppose that A is the augmented representation of a ***consistent system*** with n variables that has ***an unlimited number of solutions***.

The ***free variables*** in the ***general solution set*** correspond to the non-pivot columns of A that lie to the left of the augment line. The other variables in the system are called ***basic variables***. The value of any given basic variable might be fixed or it might be dependent upon the value(s) assigned to one or more of the free variables.

Specific solutions to the system are determined by assigning specific values to the free variables and then determining the associated values of the basic variables.

Example

State the general solution to the following system of equations and rigorously verify the solution. Also, state two specific solutions to the system.

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \qquad x_2 - x_3 + 2x_4 = -3 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right) \xrightarrow{2R_2+R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right)$$

pivot columns — free variables

$$\begin{cases} x_1 \qquad + x_4 = -4 \\ x_2 - x_3 + 2x_4 = -3 \end{cases} \Rightarrow \begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \end{cases}$$

General Solution: $\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$

Check:

$x_1 - 2x_2 + 2x_3 - 3x_4 = (-x_4-4) - 2(x_3 - 2x_4 - 3) + 2x_3 - 3x_4$
$= 2$ ✓

$x_2 - x_3 + 2x_4 = (x_3 - 2x_4 - 3) - x_3 + 2x_4$
$= -3$ ✓

A couple of specific solutions

$(-4, -3, 0, 0) \qquad (-4, -2, 1, 0)$

choices

Example
State a ***general solution*** to the following system of equations as well as two ***specific solutions*** to the system.

$$\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 6 & 5 & -2 \\ -1 & -3 & 3 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & -3 & 3 & 1 \\ 2 & 6 & 5 & -2 \end{array}\right)$$

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} -1 & -3 & 3 & 1 \\ 0 & 0 & 11 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{1}{11}R_2 \to R_2]{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 3 & -3 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{3R_2 + R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

PC PC

x_2 is free

$$\begin{cases} x_1 + 3x_2 = -1 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_2 - 1 \\ x_3 = 0 \end{cases}$$

The general solution is $\begin{cases} x_1 = -3x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$

Two specific solutions are

$(-1, 0, 0)$ and $(-4, 1, 0)$

dependent, free, fixed, fixed

$(-67, 22, 0)$

fixed

Example

Is it possible to find a value of C that makes the following system inconsistent? How do you know?

$$\begin{cases} \quad\quad 2x_2 + x_3 = -2 \\ x_1 - 4x_2 - 4x_3 = C \\ 2x_1 + 12x_2 + 5x_3 = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 1 & -4 & -4 & C \\ 2 & 12 & 5 & 7 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 2 & 12 & 5 & 7 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 20 & 13 & -2C+7 \end{array}\right)$$

$$\xrightarrow{-10R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -4 & -4 & C \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & -2C+27 \end{array}\right)$$

This cannot be made inconsistent.

$3x_3 = -2c+27 \Rightarrow x_3 = \frac{-2c+27}{3}$

& x_2 & x_1 could be found via back-substitution.

Example

Under what conditions is the following system of equations consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right) \begin{array}{c} -R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & b-a \\ 0 & -6 & 9 & 2a+c \end{array}\right)$$

$$3R_2+R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & b-a \\ 0 & 0 & 0 & -a+3b+c \end{array}\right)$$

The system will be consistent so long as $-a+3b+c=0$; i.e. $c = a-3b$

Check this out:

$$[1,1,2] - 3[1,3,-1] = [-2,-8,5]$$

I created this system by making the third row $R_1 - 3R_2$

For the system to be consistent that relationship must be maintained to the right of the augment line.