Projection vectors in $\mathbb{R}^n$

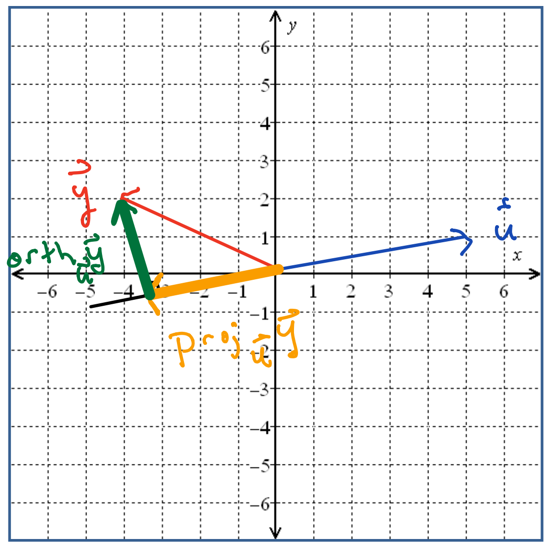
The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\operatorname{proj}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$. (vector)

The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y} - \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$. (vector)

The quantity $\operatorname{comp}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

Example 12.1

Determine and illustrate the projection vector, component vector, and scalar component when $\vec{y} = [-4, 2]^{\mathrm{T}}$ and $\vec{u} = [5, 1]^{\mathrm{T}}$.



Scalar component

$$\operatorname{comp}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} = \frac{-20+2}{25+1} = -18/26 = -9/13$$

$$\operatorname{proj}_{\vec{u}}\vec{y} = -\frac{9}{13}\vec{u} = \left\langle -\frac{45}{13}, -\frac{9}{13} \right\rangle$$

$$\boxed{\approx \langle -3.5, -0.7 \rangle}$$

$$\operatorname{orth}_{\vec{u}}\vec{y} = \vec{y} - \operatorname{proj}_{\vec{u}}\vec{y} = \langle -4, 2 \rangle - \left\langle -\frac{45}{13}, -\frac{9}{13} \right\rangle = \left\langle -\frac{7}{13}, \frac{35}{13} \right\rangle$$

$$\boxed{\approx \langle -.5, 2.7 \rangle}$$

OBSERVE: $\underbrace{\operatorname{proj}_{\vec{u}}\vec{y} + \operatorname{orth}_{\vec{u}}\vec{y}}_{\text{components (vectors) of } \vec{y} \text{ relative to } \vec{u}} = \vec{y}$

A definition and a theorem about orthogonal sets of vectors in $\mathbb{R}^n$

A set of non-zero vectors, $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i \cdot \vec{u}_j = 0 \;\forall\; i \neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Change of coordinates in $\mathbb{R}^n$

Let $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_\beta = [c_1, c_2, \ldots, c_n]^{\mathrm{T}}$ where $c_i = \text{comp}_{\vec{u}_i}(\vec{y})$.

Example 12.2

Show that the set $\beta = \{\underbrace{[1,1,1]^{\mathrm{T}}}_{\vec{v}_1}, \underbrace{[-1,-1,2]^{\mathrm{T}}}_{\vec{v}_2}, \underbrace{[3,-3,0]^{\mathrm{T}}}_{\vec{v}_3}\}$ is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. Then find $[\vec{y}]_\beta$ where $\vec{y} = [5,-2,0]^{\mathrm{T}}$.

$$\vec{v}_1 \cdot \vec{v}_2 = -1 - 1 + 2 = 0 \qquad \vec{v}_1 \cdot \vec{v}_3 = 3 - 3 + 0 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = -3 + 3 + 0 = 0$$

$\therefore$ β is indeed an orthogonal set

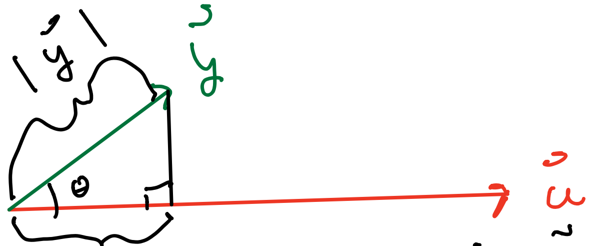
$$\text{Comp}_{\vec{v}_1}\vec{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{3}{3} = 1$$

$$\text{comp}_{\vec{v}_2}\vec{y} = \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{-3}{6} = -\frac{1}{2}$$

$$\text{Comp}_{\vec{v}_3}\vec{y} = \frac{\vec{y}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{21}{18} = \frac{7}{6}$$

$$[\vec{y}]_\beta = \begin{bmatrix} 1 \\ -1/2 \\ 7/6 \end{bmatrix}$$

Check: $(1)\begin{bmatrix}1\\1\\1\end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix}-1\\-1\\2\end{bmatrix} + \left(\frac{7}{6}\right)\begin{bmatrix}3\\-3\\0\end{bmatrix} = \begin{bmatrix}5\\-2\\0\end{bmatrix}$ ✓



Scalar component of $\vec{y}$ onto $\vec{u}$

253 definition $\rightarrow$ $\text{Comp}_{\vec{u}}\vec{y}$

SOH CAH TOA

$$\cos(\theta) = \frac{\text{comp}_{\vec{u}}\vec{y}}{|\vec{y}|}$$

$$\therefore \quad \text{comp}_{\vec{u}}\vec{y} = |\vec{y}|\cos(\theta)$$

$$\text{comp}_{\vec{u}}\vec{y} = |\vec{y}|\cos(\theta) \cdot \frac{|\vec{u}|}{|\vec{u}|}$$

$$= \frac{|\vec{u}||\vec{y}|\cos(\theta)}{|\vec{u}|}$$

$$= \frac{\vec{y}\cdot\vec{u}}{|\vec{u}|}$$

$$\therefore \quad \text{proj}_{\vec{u}}\vec{y} = \text{comp}_{\vec{u}}\vec{y} \cdot \frac{\vec{u}}{|\vec{u}|}$$

The unit vector in the same direction as $\vec{u}$

$$= \frac{\vec{y}\cdot\vec{u}}{|\vec{u}|} \cdot \frac{\vec{u}}{|\vec{u}|}$$

$$= \frac{\vec{y}\cdot\vec{u}}{|\vec{u}|^2}\,\vec{u}$$

$$= \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

yoohoo!

<u>The Gramm-Schmidt process</u> **for finding an orthogonal basis**

Let $\beta = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \}$ be a linearly independent set of vectors.

Then the set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \}$ is an orthogonal basis for the span of β where:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2) \\ \vec{v}_3 &= \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \text{proj}_{\vec{v}_1}(\vec{x}_p) - \text{proj}_{\vec{v}_2}(\vec{x}_p) - \cdots - \text{proj}_{\vec{v}_{p-1}}(\vec{x}_p) \end{aligned}$$

Furthermore, the set $\gamma = \left\{ \frac{\vec{v}_1}{|\vec{v}_1|}, \frac{\vec{v}_2}{|\vec{v}_2|}, \ldots, \frac{\vec{v}_p}{|\vec{v}_p|} \right\}$ is an ***<u>orthonormal basis</u>*** for the span of β.

Example 12.3

Let's find an orthonormal basis for the column space of $A = \begin{array}{c} \begin{array}{cccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 \end{array} \\ \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix} \end{array}$ and find the coordinates of $\vec{y} = [5, -10, -6, 7]^{\mathrm{T}}$ in terms of that basis.

orthonormal basis: $\{ \vec{v}_1, \vec{v}_2, \ldots \}$

$$\vec{v}_1 = \vec{c}_1 = \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \vec{v}_2 &= \vec{c}_2 - \text{proj}_{\vec{v}_1} \vec{c}_2 \\ &= \vec{c}_2 - \frac{\vec{c}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 \\ &= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} - \frac{-15}{30} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 9/2 \end{bmatrix} \end{aligned}$$

Let's redefine $\vec{v}_2$.

$$\begin{aligned} \vec{v}_2 &= \tfrac{2}{3} \vec{v}_{2\,\text{old}} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

check

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= 0 + 0 + 3 - 3 \\ &= 0 \ \checkmark \end{aligned}$$

$$\vec{v}_3 = \vec{C}_3 - \text{proj}_{\vec{v}_1}\vec{C}_3 - \text{proj}_{\vec{v}_2}\vec{C}_3$$

$$= \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix} - \frac{\vec{C}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{C}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix} - \frac{-75}{30}\begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \frac{15}{10}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftarrow$$ Bogus basis vector.

$\vec{0} \notin$ basis

what happened! Is the world coming to an end?

$\vec{C}_3$ is a linear combination of $\vec{C}_1$ & $\vec{C}_2$, it's redundent from a basis perspective - we lose linear independence

$$\vec{v}_3 = \vec{C}_4 - \frac{\vec{C}_4 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{C}_4 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{5}{30}\begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 1/3 \\ 2/5 \\ -2/15 \end{bmatrix}$$

Redefine $\vec{v}_3 = 15\vec{v}_3$

$$= \begin{bmatrix} 20 \\ 5 \\ 6 \\ -2 \end{bmatrix}$$

check

$\vec{v}_1 \cdot \vec{v}_3 = -40 + 20 + 18 + 2$
$= 0$ ✓

$\vec{v}_2 \cdot \vec{v}_3 = 0 + 0 + 6 - 6$
$= 0$ ✓

$$\begin{bmatrix} 1 \\ -2 \\ \bullet \\ 5 \end{bmatrix}$$

$\therefore$ our orthonormal basis:

$$\left\{ \frac{\vec{v}_1}{|\vec{v}_1|}, \frac{\vec{v}_2}{|\vec{v}_2|}, \frac{\vec{v}_3}{|\vec{v}_3|} \right\} = \left\{ \underbrace{\frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}}_{\vec{w}_2}, \underbrace{\frac{1}{\sqrt{465}}\begin{bmatrix} 20 \\ 5 \\ 6 \\ -2 \end{bmatrix}}_{\vec{w}_3} \right\}$$

Each $\vec{w}_i$ is a unit vector, so $\vec{w}_i \cdot \vec{w}_i = 1$

$$\therefore \quad \text{comp}_{\vec{w}_i} \vec{y} = \frac{\vec{y} \cdot \vec{w}_i}{\vec{w}_i \cdot \vec{w}_i}$$

$$= \vec{y} \cdot \vec{w}_i$$

$$\vec{C}_3 = \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix}$$

$$\text{comp}_{\vec{w}_1} \vec{C}_3 = \vec{C}_3 \cdot \vec{w}_1 = \frac{1}{\sqrt{30}}(-10-40-18-7) = \frac{-75}{\sqrt{70}}$$

$$\text{comp}_{\vec{w}_2} \vec{C}_3 = \vec{C}_3 \cdot \vec{w}_2 = \frac{1}{\sqrt{10}}(0+0-6+21) = \frac{15}{\sqrt{10}}$$

$$\text{comp}_{\vec{w}_3} \vec{C}_3 = \vec{C}_3 \cdot \vec{w}_3 = \frac{1}{\sqrt{465}}(100-50-36-14) = 0$$

$$\therefore \left[\vec{C}_3\right]_{\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}} = \begin{bmatrix} -75/\sqrt{30} \\ 15/\sqrt{10} \\ 0 \end{bmatrix}$$

$$\frac{-75}{\sqrt{30}} \cdot \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} + \frac{15}{\sqrt{10}} \cdot \frac{1}{\sqrt{\ }} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} + 0\,\vec{w}_3$$

$$= \begin{bmatrix} 5 \\ -10 \\ -6 \\ 7 \end{bmatrix} \checkmark$$

$$\left[\vec{C}_2\right] = \begin{bmatrix} \vec{C}_2 \cdot \vec{w}_1 \\ \vec{C}_2 \cdot \vec{w}_2 \\ \vec{C}_2 \cdot \vec{w}_3 \end{bmatrix} = \begin{bmatrix} -15/\sqrt{30} \\ 15/\sqrt{10} \\ 0 \end{bmatrix}$$

check

$$\frac{-15}{\sqrt{30}} \cdot \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -1 \end{bmatrix} + \frac{15}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} + 0\,\vec{w}_3$$

$$= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} = C_2 \quad \checkmark$$

A definition and a trio of theorems

If W is a subspace of $\mathbb{R}^n$, then the ***orthogonal compliment*** of W is the set of vectors, $W^\perp$, that are orthogonal to every vector in W.

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- $\left(\text{row}\left(A\right)\right)^\perp = \text{nul}\left(A\right)$.
- $\left(\text{col}\left(A\right)\right)^\perp = \text{nul}\left(A^{\text{T}}\right)$.

Example 12.4

Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Suppose that $\vec{u}$ and $\vec{v}$ are members of $W^\perp$.

Then, $\forall\ \vec{w} \in W$, $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$.

$$\therefore\ (\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$
$$= 0 + 0$$
$$= 0$$

($W^\perp$ is closed over vector addition)

and
$$k\vec{u} \cdot \vec{w} = k(\vec{u} \cdot \vec{w})$$
$$= k \cdot 0$$
$$= 0$$

($W^\perp$ is closed over scalar multiplication).

QED

Example 12.5

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate that $\left(\operatorname{row}(A)\right)^{\perp}=\operatorname{nul}(A)$.

If $\begin{bmatrix} x \\ y \end{bmatrix} \in \operatorname{nul}(A)$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{cases} ax+by=0 \\ cx+dy=0 \end{cases}$$

$\therefore$ $[a \ b]\cdot\begin{bmatrix} x \\ y \end{bmatrix}=0$ and $[c \ d]\cdot\begin{bmatrix} x \\ y \end{bmatrix}=0$

$\therefore$ Since $\{[a \ b], [c \ d]\}$ Spans row(A),

$\vec{v}\cdot\begin{bmatrix} x \\ y \end{bmatrix}=0 \quad \forall \vec{v} \in \operatorname{row}(A)$.

QED

Example 12.6

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate that $\left(\operatorname{col}(A)\right)^{\perp}=\operatorname{nul}\left(A^{\mathrm{T}}\right)$.

$$\left(\operatorname{col}(A)\right)^{\perp} = \left(\operatorname{row}(A^T)\right)^{\perp}$$

QED

From Ex. 12.5

$$\left(\operatorname{row}(A^T)\right)^{\perp} = \operatorname{nul}(A^T)$$

Example 12.7

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & -2 \\ -2 & 2 & -8 & 0 \\ 3 & 2 & 7 & 10 \end{bmatrix}$ and note that $A \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for each of the following vector spaces ***without actually transposing the matrix A***. In each case, write a few words so that the rationale for whatever action/conclusion you take/make is clear.

a. $\text{null}\left(A^{\mathsf{T}}\right)^{\perp}$

b. $\text{col}\left(A^{\mathsf{T}}\right)^{\perp}$

a. $\text{null}(A^T)^{\perp} = \text{row}(A^T)$

$= \text{col}(A)$

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 10 \end{bmatrix} \right\}$

b. $\text{Col}(A^T)^{\perp} = \text{row}(A)^{\perp}$

$= \text{nul}(A)$

Basis: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *unique* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$, such that $\vec{y} = \vec{w} + \vec{z}$.

In fact, if the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then

$\vec{w} + \vec{z} = \vec{y}$

$$\vec{w} = \text{proj}_{\vec{u}_1}(\vec{y}) + \text{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \text{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \vec{z} = \vec{y} - \vec{w}$$

Example 12.8

Let $\vec{u}_1 = [1 \quad -1 \quad -1 \quad 1]^{\mathrm{T}}$, $\vec{u}_2 = [2 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}$, and $\vec{u}_3 = [-1 \quad 1 \quad 1 \quad 3]^{\mathrm{T}}$. Let $W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = [2 \quad 1 \quad 3 \quad 4]$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^{\perp}$.

$$\text{Comp}_{\vec{u}_1}\vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Comp}_{\vec{u}_2}\vec{y} = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{8}{6} = \frac{4}{3}$$

$$\text{Comp}_{\vec{u}_3}\vec{y} = \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{14}{12} = \frac{7}{6}$$

$$\therefore \quad \vec{w} = \frac{3}{6}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{7}{6}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

and $\vec{z} = \vec{y} - \vec{w}$

$$= \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Check

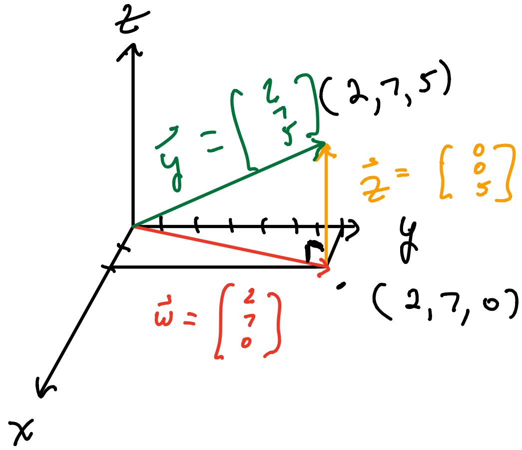
Make sure that $\vec{w} \cdot \vec{z} = 0$

$\vec{w} \cdot \vec{z} = 0 - 2 + 2 + 0$

$= 0$ ✓

Example 12.9

A basis from $\mathbb{R}^3$ for the $x_1\,x_2-\text{plane}$ is $\{\vec{u}_1, \vec{u}_2\}$ where $\vec{u}_1 = \begin{bmatrix} 3 & -4 & 0 \end{bmatrix}^T$ and $\vec{u}_2 = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}^T$. Use this basis to determine the point on the $x_1\,x_2-\text{plane}$ that is closest to the point $(2,7,5)$.



The orthogonal decomposition of $\vec{y}$ is $\vec{w} + \vec{z}$.

Obviously the point on the xy-plane closest to (2,7,5) is (2,7,0)

$$\text{comp}_{\vec{u}_1}\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{-22}{25}$$

$$\text{comp}_{\vec{u}_2}\vec{y} = \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{29}{25}$$

$$\vec{w} = -\frac{22}{25}\vec{u}_1 + \frac{29}{25}\vec{u}_2 = -\frac{22}{25}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} + \frac{29}{25}\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

oooh!

aaah!

oo, aa, special!

$$\vec{z} = \vec{y} - \vec{w} = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

check

$\vec{z}\cdot\vec{w} = 0$?

yes

Example 12.10

Let $\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find an orthonormal basis for $\text{span}\{\vec{x}_1, \vec{x}_2\}$. Next, express $\vec{x}_1$ and $\vec{x}_2$ in terms of that new basis.

Let's Gramm-Schmidt!

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}\vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{25}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 1 \\ 8/5 \end{bmatrix}$$

Check

$\vec{v}_1 \cdot \vec{v}_2 = \frac{24}{5} + 0 - \frac{24}{5} = 0$ ✓

redefine $\vec{v}_2 = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$ orthogonal basis: $\{\vec{v}_1, \vec{v}_2\}$

An orthonormal basis is

$\{\vec{w}_1, \vec{w}_2\}$ where $\vec{w}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$

& $\vec{w}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{1}{\sqrt{125}}\begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$

obviously $\vec{x}_1 = 5\vec{\omega}_1 + 0\vec{\omega}_2$

$$\text{Comp}_{\vec{\omega}_1} \vec{x}_2 = \vec{x}_2 \cdot \vec{\omega}_1$$
$$= 1$$

$$\text{Comp}_{\vec{\omega}_2} \vec{x}_2 = \vec{x}_2 \cdot \vec{\omega}_2$$
$$= \frac{25}{\sqrt{125}}$$
$$= \sqrt{5}$$

$$\therefore \quad \vec{x}_2 = 1 \cdot \vec{\omega}_1 + \sqrt{5}\,\vec{\omega}_2 \text{ ; i.e. } [\vec{x}_2]_{\{\vec{\omega}_1, \vec{\omega}_2\}} = \begin{bmatrix} 1 \\ \sqrt{5} \end{bmatrix}$$

Check

$$1 \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix} + \sqrt{5} \cdot \frac{1}{\sqrt{125}} \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \checkmark$$