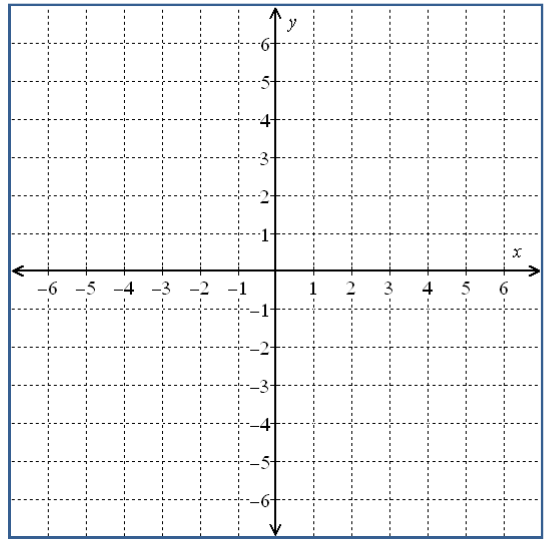
Example 12.1

Find and illustrate the projection vector, component vector, and scalar component when $\vec{y} = [2, -4]^{\mathrm{T}}$ and $\vec{u} = [5, 1]^{\mathrm{T}}$.



Example 12.2

Show that the set $\beta = \left\{ [1,1,1]^{\mathrm{T}}, [-1,-1,2]^{\mathrm{T}}, \ [3,-3,0]^{\mathrm{T}} \right\}$ is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. Then find $[\vec{y}]_\beta$ where $\vec{y} = [5, -2, 0]^{\mathrm{T}}$.

Example 12.3

Let's find an orthonormal basis for the column space of $A = \begin{bmatrix} -2 & 1 & 5 & 1 \\ 4 & -2 & -10 & 1 \\ 3 & 0 & -6 & 1 \\ -1 & 5 & 7 & 0 \end{bmatrix}$ and find the coordinates of $\vec{y} = [5, -10, -6, 7]^{\mathrm{T}}$ in terms of that basis.

Example12. 4

Prove that if W is a subspace of $\mathbb{R}^n$, then $W^{\perp}$ is also a subspace of $\mathbb{R}^n$.

Example 12.5

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate that $(\mathrm{row}(A))^{\perp} = \mathrm{nul}(A)$.

Example 12.6

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Illustrate that $(\mathrm{col}(A))^{\perp} = \mathrm{nul}(A^{\mathrm{T}})$.

Example 12.7

Let $\vec{u}_1 = [1 \ \ -1 \ \ -1 \ \ 1]^{\mathrm{T}}$, $\vec{u}_2 = [2 \ \ 1 \ \ 1 \ \ 0]^{\mathrm{T}}$, and $\vec{u}_3 = [-1 \ \ 1 \ \ 1 \ \ 3]^{\mathrm{T}}$. Let $W = \mathrm{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = [2 \ \ 1 \ \ 3 \ \ 4]^{\mathrm{T}}$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^{\perp}$.

Example 12.8

A basis from $\mathbb{R}^3$ for the $x_1\,x_2-\text{plane}$ is $\{\vec{u}_1,\vec{u}_2\}$ where $\vec{u}_1=\begin{bmatrix}3 & -4 & 0\end{bmatrix}^{\mathrm{T}}$ and $\vec{u}_2=\begin{bmatrix}4 & 3 & 0\end{bmatrix}^{\mathrm{T}}$. Use this basis to determine the point on the $x_1\,x_2-\text{plane}$ that is closest to the point $(2,7,5)$.

Example 12.9

Let $\vec{x}_1=\begin{bmatrix}4\\0\\-3\end{bmatrix}$ and $\vec{x}_2=\begin{bmatrix}2\\1\\1\end{bmatrix}$. Find an orthonormal basis for $\operatorname{span}\{\vec{x}_1,\vec{x}_2\}$. Next, express $\vec{x}_1$ and $\vec{x}_2$ in terms of that new basis.

Definitions 12.1 – 12.3: Projection vectors in $\mathbb{R}^n$

The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\operatorname{proj}_{\vec{u}}(\vec{y})=\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y}-\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The quantity $\operatorname{comp}_{\vec{u}}(\vec{y})=\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

Definition 12.4 and Theorem 12.1

A set of non-zero vectors, $\beta=\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i\cdot\vec{u}_j=0\ \forall\ i\neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Theorem 12.2: Change of coordinates in $\mathbb{R}^n$

Let $\beta=\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y}=c_1\vec{u}_1+c_2\vec{u}_2+\cdots+c_n\vec{u}_n$ where $c_i=\operatorname{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_\beta=[c_1,c_2,\ldots,c_n]^{\mathrm{T}}$ where $c_i=\operatorname{comp}_{\vec{u}_i}(\vec{y})$.

Theorem *12.3:* *<u>The Gramm-Schmidt process</u>* for finding an orthogonal basis

Let $\beta = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \}$ be a linearly independent set of vectors.

Then the set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \}$ is an orthogonal basis for the span of β where:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \operatorname{proj}_{\vec{v}_1}(\vec{x}_2) \\ \vec{v}_3 &= \vec{x}_3 - \operatorname{proj}_{\vec{v}_1}(\vec{x}_3) - \operatorname{proj}_{\vec{v}_2}(\vec{x}_3) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \operatorname{proj}_{\vec{v}_1}(\vec{x}_p) - \operatorname{proj}_{\vec{v}_2}(\vec{x}_p) - \cdots - \operatorname{proj}_{\vec{v}_{p-1}}(\vec{x}_p) \end{aligned}$$

Furthermore, the set $\gamma = \left\{ \frac{\vec{v}_1}{|\vec{v}_1|}, \frac{\vec{v}_2}{|\vec{v}_2|}, \ldots, \frac{\vec{v}_p}{|\vec{v}_p|} \right\}$ is an ***<u>orthonormal basis</u>*** for the span of β.

Definition 12.5 and Theorems 12.4 – 12.6

If W is a subspace of $\mathbb{R}^n$, then the ***<u>orthogonal compliment</u>*** of W is the set of vectors, $W^{\perp}$, that are orthogonal to every vector in W.

- $W^{\perp}$ is a subspace of $\mathbb{R}^n$.
- $(\operatorname{row}(A))^{\perp} = \operatorname{nul}(A)$.
- $(\operatorname{col}(A))^{\perp} = \operatorname{nul}(A^{\mathrm{T}})$.

Theorem 12.7: The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *<u>unique</u>* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$, such that $\vec{y} = \vec{w} + \vec{z}$.

In fact, if the set $\{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \}$ is an orthogonal basis for W, then

$$\vec{w} = \operatorname{proj}_{\vec{u}_1}(\vec{y}) + \operatorname{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \operatorname{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \vec{z} = \vec{y} - \vec{w}$$

Theorem 12.8

The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.