**Example 8.1**

Illustrate Theorem 8.1 for the span of two vectors, $\vec{u}_1$ and $\vec{u}_2$.

**Example 8.2**

Determine a basis for the set of vectors $\left\{ c_1\begin{bmatrix}2\\-2\\5\end{bmatrix}+c_2\begin{bmatrix}1\\4\\5\end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

**Example 8.3**

Determine a basis for the set of vectors $\left\{ c_1\begin{bmatrix}1\\-2\\3\end{bmatrix}+c_2\begin{bmatrix}2\\-4\\6\end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

**Example 8.4**

Determine whether or not $[5,5,10]^{\mathrm{T}} \in \operatorname{col}(A)$ where $A=\begin{bmatrix}3&4&-2\\-1&6&4\\5&14&0\end{bmatrix}$.

**Example 8.5**

Show that solutions to the equation $\begin{bmatrix}3&-7\\-6&14\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ form a subspace of $\mathbb{R}^2$ and then find a basis for that subspace.

**Example 8.6**

Let $A=\begin{bmatrix}2&-2&3\\4&1&6\end{bmatrix}$. Determine whether $\begin{bmatrix}2&1&0\end{bmatrix}^{\mathrm{T}}$ is an element of either the column space or the null space of $A$.

**Example 8.7**

Let $A=\begin{bmatrix}2&1&3&3\\1&-1&3&1\\-2&1&-5&3\\3&2&4&2\end{bmatrix}$. Then $\operatorname{RREF}(A)=\begin{bmatrix}1&0&2&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$.

a. Find a basis for the null space of $A$ and explicitly show that it is indeed a basis for the null space of $A$.
b. Find a basis for the column space of $A$ and explicitly show that it is indeed a basis for the column space of $A$.
c. State the rank of $A$.

**Example 8.8**

Let $A=\begin{bmatrix} 2 & 1 & -4 & 3 & -2 & -5 \\ -1 & 1 & 2 & 3 & 1 & -11 \\ 1 & 4 & -2 & -3 & -1 & 11 \\ -2 & -2 & 4 & -1 & 2 & -1 \end{bmatrix}$. Then $A\sim\begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. What are the dimensions of the null space and column space of $A$ and how do you know. What do these dimensions sum to?

b. State bases for the null space and column space of $A$.

**Example 8.9**

Determine the dimension of $\text{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ 0 \\ -16 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 11 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right\}\right)$.

**Example 8.10**

Find a basis for the null space of $B$ where $B=\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$.

**Definitions 8.1-8.3: Subspaces of $\mathbb{R}^n$**

A ***subspace*** of the vector space $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ that contains the zero vector from $\mathbb{R}^n$ and is closed over vector addition and scalar multiplication.

A set of vectors, $H$, is ***closed over vector addition*** if and only if $\vec{u}+\vec{v}\in H \;\forall\; \vec{u},\vec{v}\in H$.

A set of vectors, $H$, is ***closed over scalar multiplication*** if and only if $c\vec{u}\in H \;\forall\; \vec{u}\in H, c\in\mathbb{R}$.

**Theorem 8.1**

The span of a set of vectors from $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

**Definition 8.4**

A set of linearly independent vectors that span a subspace of $\mathbb{R}^n$ is called a ***basis*** for that subspace of $\mathbb{R}^n$.

**Definition 8.5 and Theorem 8.2**

The ***column space*** of a matrix $A$ is the set of all linear combinations of the column vectors of $A$. If $A$ is an $m \times n$ matrix, then $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

**Definition 8.6 and Theorem 8.3**

The ***null space*** of a matrix $A$ is the set of all solutions to the equation $A\vec{x} = \vec{0}$. If $A$ is an $m \times n$ matrix, then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

**Theorems 8.4 and 8.5: Finding bases for the null space and column space of a matrix**

- A spanning set of the solution set to the homogenous system $A\vec{x} = \vec{0}$ forms a basis for the null space of $A$.
- The pivot columns of $A$ form a basis for the column space of $A$.

**Theorems 8.6-8.8 and Definition 8.7**

If one basis of a subspace of $\mathbb{R}^n$ contains $m$ vectors, than every basis of **that** subspace contains $m$ vectors. Additionally, any set of $m$ linearly independent vectors from the subspace forms a basis for the subspace and any set of $m$ vectors that spans the subspace forms a basis for the subspace. The ***dimension*** of such a subspace is $m$, i.e. the dimension of the subspace is the number of vectors in each basis for the subspace.

**Definition 8.8 and Theorem 8.9**

The ***rank*** of a matrix $A$ is the number of nonzero rows in $\mathrm{RREF}(A)$. The dimension of the column space of $A$ is equal to the rank of $A$.

**Theorem 8.10**

If $A$ is an $m \times n$ matrix, then $\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = n$.

**Theorem 8.11: "Theorem 8" revisited**

If $A$ is an $n$ x $n$ matrix, then either each of the following statements is true about $A$ or each of the following statements is false about $A$.

a. $A$ is an invertible matrix (i.e., $A$ is ***nonsingular***).

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot columns.

d. The only solution to $A\vec{x}=\vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of $A$ form a linearly independent set.

f. The linear transformation $T(\vec{x})=A\vec{x}$ is one-to-one.

g. The equation $A\vec{x}=\vec{b}$ has exactly one solution $\forall\, \vec{b}\in\mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x})=A\vec{x}$ is onto $\mathbb{R}^n$.

l. $A^{\mathrm{T}}$ is nonsingular.

m. $\det(A)\neq 0$

n. The columns of $A$ form a basis $\mathbb{R}^n$.

o. $\mathrm{Col}(A)=\mathbb{R}^n$

p. $\dim(\mathrm{Col}(A))=n$

q. $\mathrm{Nul}(A)=\{\vec{0}\}$

r. $\dim(\mathrm{Nul}(A))=0$

**Example 9.1**

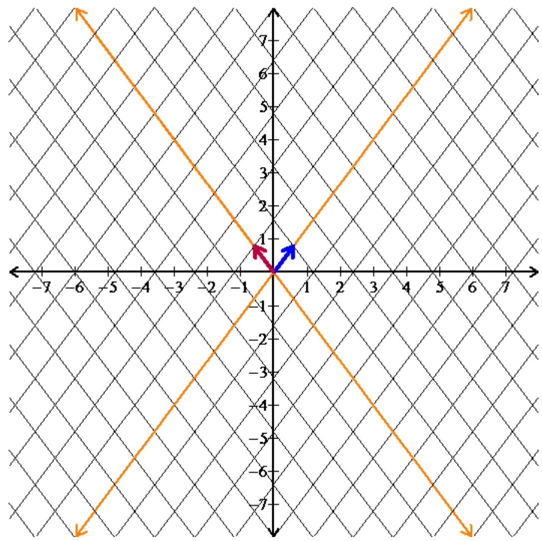
Consider the $\mathbb{R}^2$ basis $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$. Answer each of the following questions relative to $\beta$.

a. Determine $\vec{x}$ if $[\vec{x}]_\beta = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

b. Determine $[\vec{x}]_\beta$ if $\vec{x} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$.

**Example 9.2**

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{ \vec{c}_1, \vec{c}_2 \}$. Find the change-of-basis matrix from the standard basis to $\gamma$ and use that matrix to find $[\vec{x}]_\gamma$.




**Example 9.3**

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from $\beta$ to $\gamma$ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

**Theorem 9.1 and Definition 9.1**

Suppose that the set $\beta = \left\{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \right\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots c_n \vec{b}_n$. The constants $c_1 - c_n$ are called the $\beta -$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

**Theorem 9.2**

Suppose that $\beta$ and $\gamma$ are both bases for $\mathbb{R}^n$ and that $T : \mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left( [\vec{x}]_\beta \right) = [\vec{x}]_\gamma$. Then $T$ is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma \leftarrow \beta}{P}$ with the property that $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$.

**Theorem 9.3**

Suppose that $\beta$ and $\gamma$ are two ordered bases for $\mathbb{R}^n$ , $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to $\beta$ are known. Then the components of $\vec{x}$ relative to $\gamma$ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from $\beta$ to $\gamma$.

When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[ I_n \;\middle|\; \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if $\beta$ is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to $\gamma$ is simply $\gamma^{-1}$.

**Example 10.1**

Determine the eigenvalues and eigenvectors of the matrix $A$ where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

**Example 10.2**
Whence the characteristic equation?

**Example 10.3**

Determine the eigenvalues for $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$ .

**Example 10.4**

Determine bases for the eigenspaces of $B$ where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

**Example 10.5**

Determine two distinct diagonalizations of $A$ where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

**Example10.6**

Diagonalize $B$ where $B = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$; use the result to simplify $B^n$ where $n$ is a natural number. To get to the point more quickly, let's use our calculators to find the eigenvalues of $M$.

**Example 10.7**

What happens when we try to diagonalize $M$ where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$? To get to the point more expeditiously, let's use our calculators to find the eigenvalues of $M$.

**Example 10.8**
Consider the recursive series where $a_1 = 1$, $a_2 = 1$, and $a_k = 2a_{k-1} + 3a_{k-2}$ for $k \geq 3$. Let's find a general term formula (non-recursive) for $a_k$ starting at $k = 3$.

**Example 10.9**

Diagonalize $T$ where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

**Definitions 10.1-10.3: Eigenvalues and Eigenvectors (of square matrices)**

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix $A$ if there exists a scalar, $\lambda$, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar $\lambda$ is called ***an eigenvalue*** of $A$.

The eigenvalues of $A$ are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of $A$.

**Definitions 10.4 and 10.5: Eigenspaces (of square matrices)**

The set of all eigenvectors associated with the specific eigenvalue $\lambda_i$ is called the ***$\lambda_i$-eigenspace of $A$***. The dimension of the $\lambda_i$-eigenspace is called the geometric multiplicity of $\lambda_i$.

**Definition 10.6 and Theorem 10.1: Similar Matrices**

The square matrices $A$ and $B$ are ***similar matrices*** if and only if there exists a matrix $P$ with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

**NOTE**: Not all matrices that share a characteristic equation are similar!

**Theorem 10.2 and Definition 10.: 7Diagonalization of an $n \times n$ matrix *A***

If $A$ has *n* linearly independent eigenvectors, then $A$ is similar to a diagonal matrix, $D$. Furthermore, $D = P^{-1}AP$ where the columns of $P$ are composed of *n* linearly independent eigenvectors of $A$ and the main diagonal entry in the $i^{th}$ column of $D$ is the eigenvalue that corresponds to the eigenvector in the $i^{th}$ column of $P$. The product $PDP^{-1}$ is called a ***diagonalization*** of $A$.