Example 6.1

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$? Describe the set of vectors whose images are $\vec{0}$.

Example 6.2

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

Example 6.3

Draw $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$ given that T is a linear transformation and the image under T for $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are those shown in Figure 1.

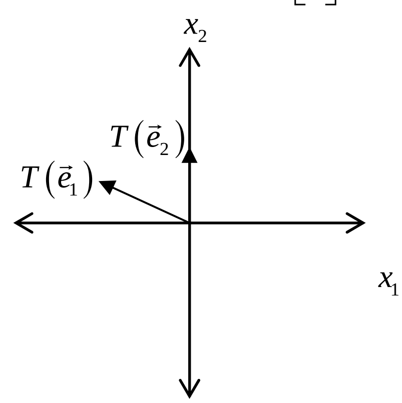

Figure 1: Transformation Vectors

Example 6.4

Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^2$ and that $T(\vec{e}_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

a. Determine $T\left(\begin{bmatrix} -6 & 2 & 1 \end{bmatrix}^{\mathrm{T}}\right)$.

b. Find a matrix, M, with the property that $T(\vec{x}) = M\,\vec{x} \;\; \forall \;\; \vec{x} \in \mathbb{R}^3$.

Example 6.5

Suppose that $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example 6.6

Show that $T(\vec{x}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}$ is a linear transformation.

Example 6.7

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the $x_1 x_2$-plane by 60^{o} in the counter-clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

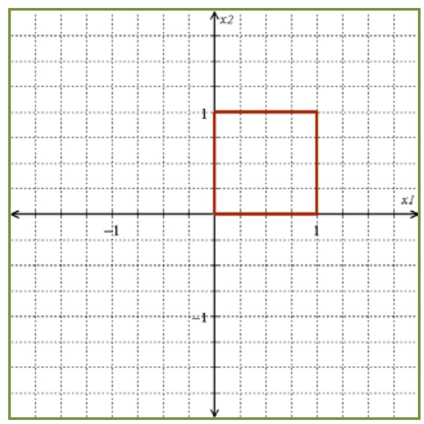

Figure 2: Rotated "unit square"

Example 6.8

Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto and also whether or not it is one-to-one.

Definitions 6.1-6.5: Transformations

A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T(\vec{x}) = \vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Definition 6.6: Linear Transformations

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(c\vec{u}) = cT(\vec{u}) \quad \forall \vec{u},\vec{v} \in \mathbb{R}^n \text{ and } c \in \mathbb{R}$$

Theorem 6.1

If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ under the linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ completely determines **all** of the images under T.

Theorem 6.2

Every transformation of form $T(\vec{x}) = A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$.

Definitions 6.7-6.8 and Theorem 6.3

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

Example 7.1

Prove that if T is a linear transformation, then $T\left(\vec{0}\right)=\vec{0}$.

Example 7.2

Prove that the linear transformation T is one-to-one if and only if the only solution to $T\left(\vec{x}\right)=\vec{0}$ is $\vec{0}$.
Hint: Prove the ***contrapositive statement***.

Example 7.3

Find all values of λ that create non-trivial solutions to the system $A\vec{x}=\vec{0}$ where A is the matrix given below. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

$$\mathrm{A} = \begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Theorem 7.1: A whole lot of equivalent properties (textbook "Theorem 8")

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x}=\vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T\left(\vec{x}\right)=A\vec{x}$ is one-to-one.

g. The equation $A\vec{x}=\vec{b}$ has exactly one solution $\forall\, \vec{b}\in\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T\left(\vec{x}\right)=A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det\left(A\right)\neq 0$