**Example 6.1**

Suppose that $T$ is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of $T$? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$ ? Describe the set of vectors whose images are $\vec{0}$ .

**Example 6.2**

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

**Example 6.3**

Draw $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$ given that $T$ is a linear transformation and the image under $T$ for $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are those shown in Figure 1.

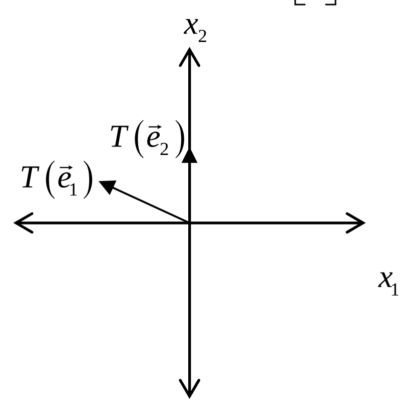



**Figure 1**: Transformation Vectors

**Example 6.4**

Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^2$ and that $T(\vec{e}_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

a. Determine $T\left(\begin{bmatrix} -6 & 2 & 1 \end{bmatrix}^{\mathrm{T}}\right)$ .

b. Find a matrix, $M$, with the property that $T(\vec{x}) = M\,\vec{x} \;\; \forall \;\; \vec{x} \in \mathbb{R}^3$.

**Example 6.5**

Suppose that $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix for $T$ if $T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

**Example 6.6**

Suppose that $T_1:\mathbb{R}^2 \to \mathbb{R}^2$ by reflecting vectors from $\mathbb{R}^2$ across the line $y = \sqrt{3}\,x$ and that $T_2:\mathbb{R}^2 \to \mathbb{R}^2$ by rotating vectors from $\mathbb{R}^2$ $30^{\circ}$ in the counter-clockwise direction. Suppose further that $T:\mathbb{R}^2 \to \mathbb{R}^2$ by the rule $T(\vec{x}) = T_2(T_1(\vec{x}))$. Determine the linear transformation matrix for $T$ (i.e. determine the matrix $A$ such that $T(\vec{x}) = A\vec{x}\ \forall\ \vec{x} \in \mathbb{R}^2$). Finally, confirm the matrix $A$ by tracking the vector $\vec{x} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ under $T$ and then computing $A\vec{x}$.

**Example 6.7**

Determine a general matrix for the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors from $\mathbb{R}^2$ through an angle $\theta$.

**Example 6.8**

Find a matrix $A$ with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the $x_1\,x_2$-plane by $60^{\circ}$ in the clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

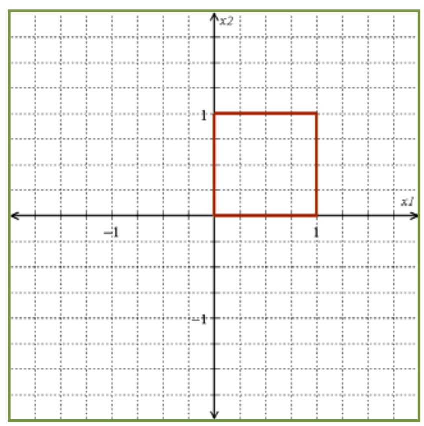



**Figure 2**: Rotated "unit square"

**Example 6.9**

Determine the linear transformation matrix for $T$ given that $T\left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ -4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

**Definitions 6.1-6.5: Transformations**

A ***transformation***, $T$, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T\left(\vec{x}\right)=\vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under $T$.

$\mathbb{R}^n$ is called the ***domain*** of $T$ and $\mathbb{R}^m$ is called the ***codomain*** of $T$. The set of all images found under $T$ is called the ***range*** of $T$.

**Definition 6.6: Linear Transformations**

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T\left(\vec{u}+\vec{v}\right)=T\left(\vec{u}\right)+T\left(\vec{v}\right) \text{ and } T\left(c\vec{u}\right)=cT\left(\vec{u}\right) \quad \forall \vec{u},\vec{v}\in\mathbb{R}^n \text{ and } c\in\mathbb{R}$$

**Theorem 6.1**

If we let $\vec{e}_i$ represent the $i^{\text{th}}$ column of $I_n$, then the images of $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n$ under the linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ completely determines **all** of the images under $T$.

**Theorem 6.2**

Every transformation of form $T\left(\vec{x}\right)=A\vec{x}$ is a linear transformation and if $T$ is a linear transformation there exists a unique matrix $A$ such that $T\left(\vec{x}\right)=A\vec{x}$.

**Example 7.1**
Categorize each of the following function as one-to-one (injective), onto $\mathbb{R}$ (surjective), or both one-to-one and onto $\mathbb{R}$ (bijective): $y=\tan(x)$, $y=\sin(x)$, $y=\tan^{-1}(x)$, $y=x^3$.

**Example 7.2**

Let $A=\begin{bmatrix}3 & -2 & -16\\ 2 & 4 & 16\end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x})=A\vec{x}$ is onto $\mathbb{R}^2$ and also whether or not it is one-to-one

**Example 7.3**

Prove that if $T$ is a linear transformation, then $T(\vec{0})=\vec{0}$.

**Example 7.4**

Prove that the linear transformation $T$ is one-to-one if and only if the only solution to $T(\vec{x})=\vec{0}$ is $\vec{0}$.
**Hint**: Prove the ***contrapositive statement***.

**Example 7.5:** Let $\mathrm{A}=\begin{bmatrix}1 & \lambda & 0\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$

Suppose that $T(\vec{x})=A\vec{x}$. Find all values of $\lambda$ that make $T$ one-to-one. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

**Example 7.6**
Demonstrate, both implicitly and explicitly, that the linear transformation T, given below, is neither one-to-one nor onto $\mathbb{R}^3$.

$$T\left(\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\right)=\begin{bmatrix}-2x_1+x_3\\ x_1+x_2\\ 3x_1+x_2-x_3\end{bmatrix}$$

**Definitions and a Theorem 7.1**

The linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

The linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u})=T(\vec{v}) \Leftrightarrow \vec{u}=\vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x})=\vec{0}$ is $\vec{0}$.

**Theorem 7.2: A whole lot of equivalent properties (textbook "Theorem 8")**

If $A$ is an $n$ x $n$ matrix, then either each of the following statements is true about $A$ or each of the following statements is false about $A$.

a. $A$ is an invertible matrix (i.e., $A$ is ***nonsingular***).

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of $A$ form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall\ \vec{b} \in \mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. $A^{\mathrm{T}}$ is nonsingular.

m. $\det(A) \neq 0$

**Example 8.1**

Illustrate Theorem 8.1 for the span of two vectors, $\vec{u}_1$ and $\vec{u}_2$.

**Example 8.2**

Determine a basis for the set of vectors $\left\{ c_1 \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

**Example 8.3**

Determine a basis for the set of vectors $\left\{ c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

**Example 8.4**

Determine whether or not $[5,5,10]^{\mathrm{T}} \in \operatorname{col}(A)$ where $A = \begin{bmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{bmatrix}$.

**Example 8.5**

Show that solutions to the equation $\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form a subspace of $\mathbb{R}^2$ and then find a basis for that subspace.

**Example 8.6**

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & 6 \end{bmatrix}$. Determine whether $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$ is an element of the column space or the null space of $A$.

**Example 8.7:** Let $A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\operatorname{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Find a basis for the null space of $A$ and explicitly show that it is indeed a basis for the null space of $A$.
b. Find a basis for the column space of $A$ and explicitly show that it is indeed a basis for the column space of $A$.
c. State the rank of $A$.
d. State a basis for the row space of $A$.
e. State each row of $A$ as a linear combination of the basis vectors stated in part (d).

**Example 8.8**

Let $A=\begin{bmatrix} 2 & 1 & -4 & 3 & -2 & -5 \\ -1 & 1 & 2 & 3 & 1 & -11 \\ 1 & 4 & -2 & -3 & -1 & 11 \\ -2 & -2 & 4 & -1 & 2 & -1 \end{bmatrix}$. Then $A\sim\begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. What are the dimensions of the null space, column space, and row space of $A$ and how do you know. What does the null-space dimension and column space dimension sum to?

b. State bases for the null space, column space, and row space of $A$.

**Example 8.9**

Determine the dimension of $\text{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ 0 \\ -16 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 11 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right\}\right)$.

**Example 8.10**

Let $A=\begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$. The $A\sim\begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Fill-in each of the following blanks.

The column space of $M$ is a ________________-dimensional subspace of $\mathbb{R}^{\_\_\_}$.

The row space of $M$ is a ________________-dimensional subspace of $\mathbb{R}^{\_\_\_}$.

The null space of $M$ is a ________________-dimensional subspace of $\mathbb{R}^{\_\_\_}$.

The column space of $M^{\mathrm{T}}$ is a ________________-dimensional subspace of $\mathbb{R}^{\_\_\_}$.

The row space of $M^{\mathrm{T}}$ is a ________________-dimensional subspace of $\mathbb{R}^{\_\_\_}$.

The null space of $M^{\mathrm{T}}$ is a ________________-dimensional subspace of $\mathbb{R}^{\_\_\_}$.

**Definitions 8.1-8.3: Subspaces of $\mathbb{R}^n$**

A ***subspace*** of the vector space $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ that contains the zero vector from $\mathbb{R}^n$ and is closed over vector addition and scalar multiplication.

A set of vectors, $H$, is ***closed over vector addition*** if and only if $\vec{u}+\vec{v}\in H\ \forall\ \vec{u},\vec{v}\in H$ .

A set of vectors, $H$, is ***closed over scalar multiplication*** if and only if $c\vec{u}\in H\ \forall\ \vec{u}\in H, c\in\mathbb{R}$ .

**Theorem 8.1**

The span of a set of vectors from $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ .

**Definition 8.4**

A set of linearly independent vectors that span a subspace of $\mathbb{R}^n$ is called a ***basis*** for that subspace of $\mathbb{R}^n$ .

**Definition 8.5 and Theorems 8.2-8.3**

The ***column space*** of a matrix $A$ is the set of all linear combinations of the column vectors of $A$. If $A$ is an $m\times n$ matrix, then $\mathrm{col}(A)$ is a subspace of $\mathbb{R}^m$. The pivot columns of $A$ form a basis for the column space of $A$.

**Definition 8.6 and Theorems 8.4-8.5**

The ***null space*** of a matrix $A$ is the set of all solutions to the equation $A\vec{x}=\vec{0}$. If $A$ is an $m\times n$ matrix, then $\mathrm{nul}(A)$ is a subspace of $\mathbb{R}^n$. A spanning set of the solution set to the homogenous system $A\vec{x}=\vec{0}$ forms a basis for the null space of $A$.

**Definition 8.7 and Theorems 8.6-8.7**

The ***row space*** of a matrix $A$ is the set of all linear combinations of the row vectors of $A$. If $A$ is an $m\times n$ matrix, then $\mathrm{row}(A)$ is a subspace of $\mathbb{R}^n$. The pivot columns of $A$ form a basis for the column space of $A$. The non-zero rows of $\mathrm{RREF}(A)$ for a basis for the row space of $A$.

**Theorems 8.8-8.10 and Definition 8.8**

If one basis of a subspace of $\mathbb{R}^n$ contains $m$ vectors, than every basis of **that** subspace contains $m$ vectors. Additionally, any set of $m$ linearly independent vectors from the subspace forms a basis for the subspace and any set of $m$ vectors that spans the subspace forms a basis for the subspace. The ***dimension*** of such a subspace is $m$, i.e. the dimension of the subspace is the number of vectors in each basis for the subspace.

**Theorem 8.11: "Theorem 8" revisited**

If $A$ is an $n$ x $n$ matrix, then either each of the following statements is true about $A$ or each of the following statements is false about $A$.

a. $A$ is an invertible matrix (i.e., $A$ is ***nonsingular***).

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of $A$ form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall\, \vec{b} \in \mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. $A^{\mathrm{T}}$ is nonsingular.

m. $\det(A) \neq 0$

n. The columns of $A$ form a basis $\mathbb{R}^n$.

o. $\mathrm{Col}(A) = \mathbb{R}^n$

p. $\dim(\mathrm{Col}(A)) = n$

q. $\mathrm{Nul}(A) = \{\vec{0}\}$

r. $\dim(\mathrm{Nul}(A)) = 0$

**Example 9.1**

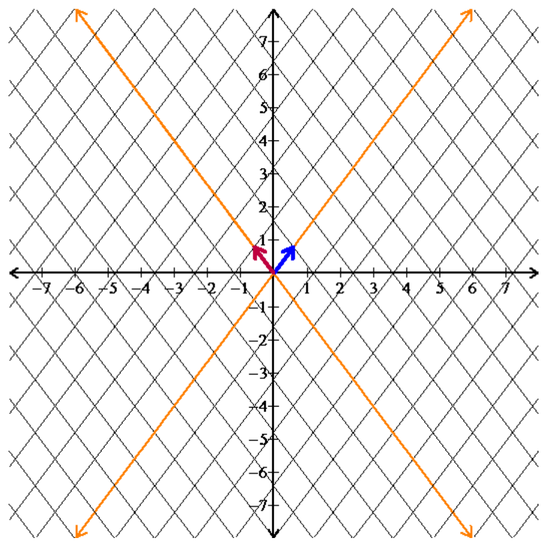
Consider the $\mathbb{R}^2$ basis $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$. Answer each of the following questions relative to $\beta$.

a. Determine $\vec{x}$ if $[\vec{x}]_\beta = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

b. Determine $[\vec{x}]_\beta$ if $\vec{x} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$.

**Example 9.2**

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{ \vec{c}_1, \vec{c}_2 \}$. Find the change-of-basis matrix from the standard basis to $\gamma$ and use that matrix to find $[\vec{x}]_\gamma$.




**Example 9.3**

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from $\beta$ to $\gamma$ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

**Theorem 9.1 and Definition 9.1**

Suppose that the set $\beta = \left\{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \right\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots c_n \vec{b}_n$. The constants $c_1 - c_n$ are called the $\beta -$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

**Theorem 9.2**

Suppose that $\beta$ and $\gamma$ are both bases for $\mathbb{R}^n$ and that $T : \mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left( [\vec{x}]_\beta \right) = [\vec{x}]_\gamma$. Then $T$ is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma \leftarrow \beta}{P}$ with the property that $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$.

**Theorem 9.3**

Suppose that $\beta$ and $\gamma$ are two ordered bases for $\mathbb{R}^n$ , $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to $\beta$ are known. Then the components of $\vec{x}$ relative to $\gamma$ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from $\beta$ to $\gamma$.

When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[ I_n \mid \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if $\beta$ is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to $\gamma$ is simply $\gamma^{-1}$.

**Example 10.1**

Determine the eigenvalues and eigenvectors of the matrix $A$ where $A=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

**Example 10.2**
Whence the characteristic equation?

**Example 10.3**

Determine the eigenvalues for $A=\begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$ .

**Example 10.4**

Determine bases for the eigenspaces of $B$ where $B=\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

**Example 10.5**

Determine two distinct diagonalizations of $A$ where $A=\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

**Example10.6**

Diagonalize $B$ where $B=\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$; use the result to simplify $B^n$ where $n$ is a natural number. To get to the point more quickly, let's use our calculators to find the eigenvalues of $M$.

**Example 10.7**

What happens when we try to diagonalize $M$ where $M=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$? To get to the point more expeditiously, let's use our calculators to find the eigenvalues of $M$.

**Example 10.8**
Consider the recursive series where $a_1=1$, $a_2=1$, and $a_k=2a_{k-1}+3a_{k-2}$ for $k\geq 3$. Let's find a general term formula (non-recursive) for $a_k$ starting at $k=3$.

**Example 10.9**

Diagonalize $T$ where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

**Example 10.10**

Explain ***geometrically*** why the rotation matrix $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ cannot possibly have any real number eigenvalues for $0 < \theta < \pi$.

**Definitions 10.1-10.3: Eigenvalues and Eigenvectors (of square matrices)**

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix $A$ if there exists a scalar, $\lambda$, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar $\lambda$ is called ***an eigenvalue*** of $A$.

The eigenvalues of $A$ are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of $A$.

**Definitions 10.4 and 10.5: Eigenspaces (of square matrices)**

The set of all eigenvectors associated with the specific eigenvalue $\lambda_i$ is called the ***$\lambda_i$-eigenspace of $A$***. The dimension of the $\lambda_i$-eigenspace is called the geometric multiplicity of $\lambda_i$.

**Definition 10.6 and Theorem 10.1: Similar Matrices**

The square matrices $A$ and $B$ are ***similar matrices*** if and only if there exists a matrix $P$ with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

**NOTE**: Not all matrices that share a characteristic equation are similar!

**Theorem 10.2 and Definition 10.: 7Diagonalization of an $n \times n$ matrix *A***

If $A$ has $n$ linearly independent eigenvectors, then $A$ is similar to a diagonal matrix, $D$. Furthermore, $D = P^{-1}AP$ where the columns of $P$ are composed of $n$ linearly independent eigenvectors of $A$ and the main diagonal entry in the $i^{\text{th}}$ column of $D$ is the eigenvalue that corresponds to the eigenvector in the $i^{\text{th}}$ column of $P$. The product $PDP^{-1}$ is called a ***diagonalization*** of $A$.