**Example 1.1**

Use an augmented matrix to mimic the elimination method for solving the following linear system of equations.

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}.$$

**Example 1.2**

Use the method of Gaussian elimination to find an echelon form of the augmented matrix representation for each of the following systems of equations and use that matrix to determine the solution to the system of equations.

a. $\begin{cases} 2x_1 - 5x_2 - 3x_3 = -23 \\ -5x_1 + x_2 - 2x_3 = -7 \\ x_1 + 3x_2 + x_3 = 3 \end{cases}$

b. $\begin{cases} x_1 - 2x_2 + x_4 = 4 \\ -2x_1 + 3x_2 - 2x_3 + 2x_4 = -3 \\ -x_2 - 2x_3 = -11 \\ 5x_1 - 10x_2 - 3x_4 = -1 \end{cases}$

**Example 1.3**

Several augmented row echelon form matrices are given below (and on the next page). For each matrix, identify the pivot columns and state the nature of the solution set for the associated system of equations.

a. $\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b. $\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$

c. $\mathbf{C} = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

**Example 1.4**

Solve the following systems of equations using the Gauss-Jordan elimination method.

a. $\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$

b. $\begin{cases} 2x_2 - 6x_3 = -2 \\ 4x_1 - x_2 + 3x_3 = 1 \\ -x_1 + 3x_2 - 8x_3 = -4 \end{cases}$

c. $\begin{cases} -x_1 + 6x_2 - 2x_3 = 9 \\ 3x_1 - 2x_2 + x_3 + 5x_4 = -1 \\ 2x_1 + 4x_2 - x_3 + 5x_4 = 8 \\ -3x_1 - x_2 + x_3 - 7x_4 = -6 \end{cases}$

**Example 1.5**

State a ***general solution*** to the following systems of equations as well as two ***specific solutions*** to the system.

a. $\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \phantom{x_1 - 2} x_2 - x_3 + 2x_4 = -3 \end{cases}$ b. $\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$

**Example 1.6**

Under what conditions is the following system of equations consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$$

**Example 1.7: Application: Network Analysis**

A network is most easily thought of as a city street system. The intersections are technically called ***nodes*** or ***junctions*** and each directed stretch of road between intersections is called a ***branch***. Because branches are directed, if there is a two-way street between two intersections the corresponding network will have two branches between the corresponding nodes. We assign values or variables to each branch; those values and variables could conceptually represent flow-rates or flow-amounts along those branches. In order for the network to be valid, **the total flow into the network must equal the total flow out of the network**. The values and variables in Figure 1 represent traffic flow rates (vehicles/quarter-hour) in a small section of a city street system. Let's determine the minimum and maximum flow rates through each of the variable branches.

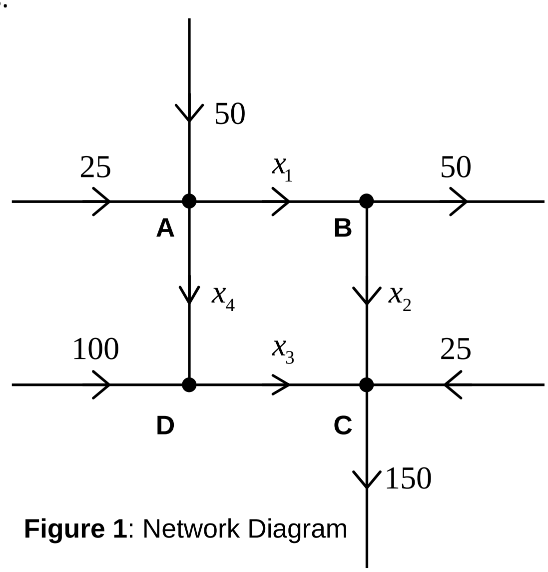



**Figure 1**: Network Diagram

**Definitions 1.1-1.5**

The Gaussian elimination process for solving systems of linear equations is predicated upon the three ***elementary row operations***. The three elementary row operations are:

1. Interchanging two rows of the matrix
2. Replacing one row of the matrix with a non-zero multiple of itself
3. Replacing one row of a matrix with itself added to a multiple of another row of the matrix

Two matrices are said to be ***row equivalent*** if one matrix can be transformed into the other via a series of elementary row operations. The first non-zero entry of any row of a matrix is called ***the leading entry*** of that row. A matrix is said to be in **(*row*) *echelon form*** if the matrix satisfies both of the following properties.

*i*. Every row that contains nothing but zeros occurs at the bottom of the matrix.
*ii*. The leading entry of any non-zero row appears to the right of the leading entry in the row directly above it.

The process of transforming a matrix into a row equivalent matrix of echelon form is called the ***Gaussian elimination process***.

## Definitions 1.6-1.9 and Theorem 1.1

A ***pivot position*** in a matrix is an entry position that corresponds to a leading entry in a row echelon form of the matrix. Any column that contains a pivot entry is called a ***pivot column***.

If $\mathrm{A}$ is an augmented matrix representing a system of equation with $n$ variables, then:

- the system of equations has exactly one solution if and only if $\mathrm{A}$ has exactly $n$ pivot columns all of which lie to the left of the augment line;
- the system of equations has no solutions if and only if the right-most column of $\mathrm{A}$ is a pivot column;
- the system of equations has an unlimited number of solutions if and only if there are less than $n$ pivot columns in $\mathrm{A}$ and the right-most column of $\mathrm{A}$ is not a pivot column.

A system of equations with at least one solution is called ***consistent***. A system of equations with no solutions is called ***inconsistent***.

## Gauss-Jordan Elimination Method

The ***Gauss-Jordan elimination method*** follows the same row manipulation rules as Gaussian elimination. What distinguishes the two methods is that the Gaussian method stops once you've reduced the matrix to row echelon form where-as the Gauss-Jordanian method requires you to further reduce the matrix to ***reduced (row) echelon form***. That is, the simplified matrix must meet each of the following properties:

*i*. Every row that contains nothing but zeros occurs at the bottom of the matrix.
*ii*. The leading entry of any non-zero row appears to the right of the leading entry in the row directly above it.
*iii*. All entries directly above or below a leading entry are zero.
*iv*. Every leading entry is 1.

When manipulating a matrix into reduced row echelon form, there are essentially three tasks that need to be completed. In the list below, you always want to complete task A first. Tasks B and C can be done in either order.

A. Manipulate every entry below a leading entry to zero. This task should be worked left-to-right; i.e., first create the necessary zeros in the first column, then the second column, then the third column, etc. Remember that the leading entries need to move rightward as you move down the rows of the matrix. Occasionally you will need to swap rows to maintain this cascading effect. **Always use the row containing the leading entry to create the zeros below that leading entry**
B. Manipulate every entry above a leading entry to zero. This task should be worked right-to-left. **Always use the row containing the leading entry to create the zeros above that leading entry.**
C. Multiply each of the rows by the necessary constants so that each leading entry is 1.

## Definitions 1.10-1.13

Suppose that $\mathrm{A}$ is the augmented representation of a ***consistent system*** with *n* variables that has ***an unlimited number of solutions***.

The ***free variables*** in the ***general solution set*** correspond to the non-pivot columns of $\mathrm{A}$ that lie to the left of the augment line. The other variables in the system are called ***basic variables***. The value of any given basic variable might be fixed or it might be dependent upon the value(s) assigned to one or more of the free variables.

***Specific solutions*** to the system are determined by assigning specific values to the free variables and then determining the associated values of the basic variables.

**Example 2.1**

Simplify $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$ and illustrate the process on Figure 1.

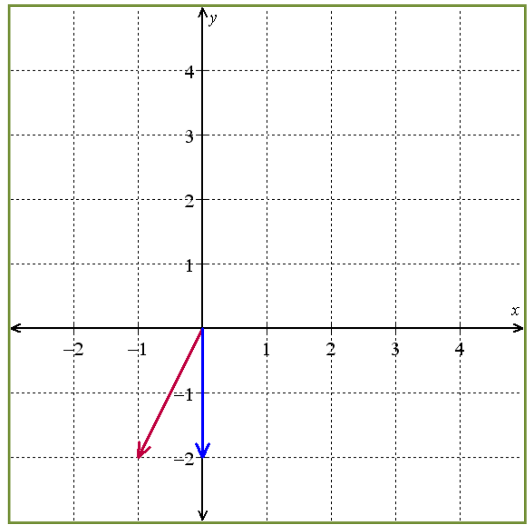



**Figure 1**: $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$

**Example 2.2**

Let $\vec{a}_1=\begin{bmatrix}2\\-4\end{bmatrix}$, $\vec{a}_2=\begin{bmatrix}-5\\-1\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}12\\-13\end{bmatrix}$. Express $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

**Example 2.3**

Let $A=\begin{bmatrix}3&-1&5\\-2&0&-4\\-1&4&2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}3\\h\\2\end{bmatrix}$. Find the value of $h$ if $\vec{b}$ is in the span of the columns of $A$.

**Example 2.4**

Let $A=\begin{bmatrix}2&-6&-1\\0&5&2\end{bmatrix}$, $B=\begin{bmatrix}1&-3\\4&3\\-2&7\end{bmatrix}$, $\vec{u}=\begin{bmatrix}10\\2\end{bmatrix}$, and $\vec{v}=\begin{bmatrix}-2\\4\\8\end{bmatrix}$. Find each of the following products (where possible): $A\vec{u}$, $A\vec{v}$, $B\vec{u}$, and $B\vec{v}$.

**Example 2.5**

Write $\begin{bmatrix}-1&2&5\\8&-2&0\\1&-3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7\\11\\-3\end{bmatrix}$ as a vector equation.

**Example 2.6**

Write the system $\begin{cases}2x_1 - 3x_2 + x_3 - 2x_4 = 0\\ 5x_1 - x_2 + x_4 = 9\end{cases}$ as a matrix equation of form $A\vec{x}=\vec{b}$.

**Example 2.7: Application: Balancing Chemical Equations**

Ethane and Oxygen combine to produce Carbon Dioxide and steam. Formally, this is represented by the equation $C_2H_6 + O_2 \to CO_2 + H_2O$. Let's put our newfound skills to use and balance this equation. Speaking of putting things to use … let's use our calculator to find the RREF form of the matrix.

**Definition 2.7: Application: Center of Mass**

The center of mass (or center of gravity) of an object is the average of the product of mass points in the object with their relative distance from a fixed reference point. The concept is most easily understood if on thinks about a teeter-totter. When two people of different mass sit on a teeter totter of uniform density and thickness, the center of mass is clearly going to be closer to the heavier person than the lighter person; this phenomenon is reflected in the fact that if the teeter-totter is to stay if balance, the heavier person needs to sit closer to the tipping-point than the lighter person. Unfortunately, we can't define the center of mass in all cases as the balance point, because the center of mass is frequently not even on the object! This is easy to see if you think about a washer (as in nuts and bolts) of uniform density; the center of mass is clearly the center point of the hole in the middle of the washer.

For most objects, it takes a double or triple integral to calculate the center of mass. In some simple situations, however, the point can be determined by a simple formula. For example, if we have a triangular lamina of uniform mass-density referenced to the $xy$-plane, the center of mass ($\vec{v}$) can be determined using the formula $\vec{v} = \frac{1}{3}\left[\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right]$ where $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ represent the vectors from the origin to the three vertices of the triangle; for ease of reference, we shall contextually refer to these vectors as points.

That formula is a special case of the more general formula for the center of mass of several mass points.

**Example 2.8**

Find the center of mass of the triangular lamina outlined in Figure 2 assuming that the lamina has uniform density and thickness.

**Example 2.9**

In Example 2.8 we found the center of mass using the formula $\vec{v} = \frac{1}{m}\sum_{i=1}^{n} m_i v_i$ and assuming that there was 1 g of mass at each of the vertices. Suppose that we had 9 addition grams of mass to distribute among the vertices. How should the mass be distributed so that the center of mass of the lamina shifts to the point $(3, -2)$ ?

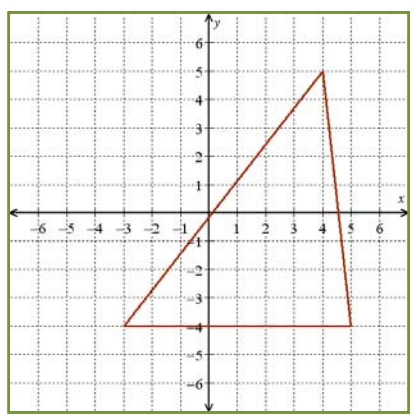



**Figure 2**: lamina of uniform density and thickness

**Definitions 2.1-2.3**

A ***column vector*** is a matrix with only one column. Assuming that we limit the entries to real numbers, the set of all 2 x 1 (column) vectors is called $\mathbb{R}^2$ and the set of all 3 x 1 vectors is called $\mathbb{R}^3$ .

***Scalar multiplication*** is the process of multiplying a vector by a real number; the process is effected by multiplying each entry in the vector by the scalar.

***Vector addition*** and ***vector subtraction*** are effected by adding or subtracting the corresponding entries of the two vectors; both of these operations can only be preformed between vectors with the same number of rows.

**Definitions 2.4 and 2.5**

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$ . We say that $\vec{y}$ is a ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ if and only if there exist scalars, $c_1, c_2, \ldots, c_p$ such that $\vec{y} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots c_p \vec{v}_p$ . If such scalars exist, they are called the ***weights*** in the linear combination.

**Definition 2.6**

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all vectors in $\mathbb{R}^n$ . The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the ***span*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ .

**Theorem 2.1**

The center of mass ( $\vec{v}$ ) of $n$ mass points is: $\vec{v} = \dfrac{1}{m}\sum_{i=1}^{n} m_i v_i$ where $m_i$ is the mass at point $v_i$ and $m$ is the sum of all the masses.

**Notes 2.1 and 2.2**

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ can be expressed as a linear combination of the columns of $A$; that is, $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is in the span of the columns of $A$.
Later in the term the span of the columns of $A$ will be defined as the ***column space*** of $A$. It follows that the equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is in the column space of $A$.

**Definition 2.8**

The product of an $m$ x $n$ matrix, $A$, and $n$ x $1$ vector, $\vec{x}$ , is defined by $A\vec{x} = \sum_{i=1}^{n} x_i \vec{a}_i$ where $x_i$ is the entry in the $i^{\text{th}}$ row of $\vec{x}$ and $\vec{a}_i$ is the $i^{\text{th}}$ column of $A$ (treating the columns of $A$ as vectors). You cannot find the product $A\vec{x}$ unless the number of columns of $A$ is equal to the number of rows of $\vec{x}$ .

**Example 3.1**

Determine whether or not each of the following is a homogenous system of equations.

a. $\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$

b. $4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**c.** 

**Example 3.2**

Describe the solution set to the homogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

**Example 3.3**

Compare the solution set from the last example with the solution set to the nonhomogenous system $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}$.

**Example 3.4**

Compare and contrast the solution sets to the homogenous system $\{3x_1 - 12x_2 - 6x_3 = 0$ and the nonhomogenous system $\{3x_1 - 12x_2 - 6x_3 = -15$.

**Example 3.5**

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

**Example 3.6**

Determine the values of $a$ that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

**Example 3.7**

Prove Theorem 3.3.

**Example 3.8**

Determine whether or not the columns of $A$ span $\mathbb{R}^3$ where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$.

**Example 3.9**

Consider the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. Explain why the set cannot possibly span $\mathbb{R}^3$. Afterwards, add a vector to the set so that it does span $\mathbb{R}^3$.

**Example 3.10**

Determine which of the following sets are linearly independent; explain!

a. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$

**Definitions 3.1-3.3 and Theorem 3.1**

A ***homogeneous system of equations*** is a system that can be written in the form $A\vec{x} = \vec{0}$ .

Every homogeneous system of equations has at least one solution ( $\vec{0}$ ); $\vec{0}$ is called the ***trivial solution*** to a homogeneous system of equations. Any other solution to a homogenous system of equations is called a ***nontrivial solution***.

**Definitions 3.4 and 3.5**

The set of vectors $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ is said to be ***linearly independent*** if and only if the only solution to the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}$ is $c_1 = c_2 = \cdots c_n = 0$. The set is said to be ***linearly dependent*** if there is a nontrivial solution to the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}$.

**Theorems 3.2-3.5**

- A set of vectors containing more than $n$ vectors from $\mathbb{R}^n$ must be linearly dependent.
- A set of **two** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector.
- A set of $n$ vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent.
- If $A$ is an $m$ x $n$ matrix, then the columns of $A$ span $\mathbb{R}^m$ if and only if $A$ has a pivot position in every row.

**Example 4.1**

Consider the matrix $B = \begin{bmatrix} 4 & 7 & -7 \\ 8 & -2 & 0 \\ -4 & 11 & 9 \\ -18 & -62 & 3 \\ 22 & -1 & 14 \end{bmatrix}$.

a. What are the dimensions of the matrix?

b. What are entries $b_{32}$, $b_{23}$, $b_{41}$, and $b_{14}$?

**Example 4.2**

Categorize each of the following as a vector, a column vector, or not a vector.

i. $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 7 \end{bmatrix}$ ii. $\begin{bmatrix} 3 & -1 \\ 6 & 12 \end{bmatrix}$ iii. $[4, 7, -18]$ iv. $[5 \quad 0]$ v. 

**Example 4.3**

Consider the matrices $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \\ 0 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 8 & -2 \\ -5 & -1 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & -2 & 6 \\ 3 & -2 & -2 \end{bmatrix}$.

Find $A + B$, $B - 2C$, and $C + A^{\mathrm{T}}$

**Example 4.4**

Which of the following are column vectors?

i. $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$ ii. $\begin{bmatrix} 3 & -1 \\ 6 & 12 \end{bmatrix}$ iii. $[4, 7, -18]^{\mathrm{T}}$ iv. $[5 \quad 0]$ v. 

**Example 4.5**

Find $AB$ if $A = \begin{bmatrix} 2 & 1 \\ 4 & 6 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 0 & 7 \\ -3 & 5 & -4 & -5 \end{bmatrix}$.

**Example 4.6**

Find $AB$ if $A=\begin{bmatrix} 2 & 1 & 0 & 8 & -5 \\ 1 & 3 & 4 & -3 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -2 \\ 10 & 1 \\ 2 & 4 \\ 0 & 3 \\ -6 & 7 \end{bmatrix}$.

**Example 4.7**

Find $AB$ if $A=\begin{bmatrix} 2 & 1 & 0 & 8 & -5 \\ 1 & 3 & 4 & -3 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -2 \\ 10 & 1 \\ 2 & 4 \end{bmatrix}$.

**Example 4.8**

Write a simple dimensional proof that establishes that in general, matrix multiplication is not commutative; i.e., in general $AB \neq BA$.

**Example 4.9**

Use your calculator to find $I_3 A$ and $AI_3$ where $A=\begin{bmatrix} 2 & 6 & -3 \\ -1 & 5 & 4 \\ 9 & 12 & -7 \end{bmatrix}$ and $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Example 4.10**

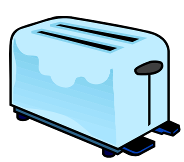
Use your calculator to find $AB$ where $A=\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}$.

**Example 4.11**

Which of the following is $\mathbf{C}^{-1}$, where $\mathbf{C}=\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$?

a. $\mathbf{A}=\begin{bmatrix} 1/5 & 1/3 \\ 1/2 & 1 \end{bmatrix}$

b. $\mathbf{B}=\begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$

c. 

d. $\mathbf{D}=\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$

**Example 4.12**

Determine – **by hand** - if they exist, the inverses of each of the following matrices.

$$A=\begin{bmatrix}-12 & -5 & -9\\ -4 & -2 & -4\\ -8 & -4 & -6\end{bmatrix},\ B=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}\text{ and } C=\begin{bmatrix}7 & -12\\ 18 & 4\\ 0 & -9\end{bmatrix}$$

**Example 4.13**

Determine a general formula for the inverse of the matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

**Example 4.14**

Let's use the formula derived in Example 4.13 to help us solve the system $\begin{cases}4x_1 + 3x_2 = 20\\ -2x_1 + 5x_2 = -36\end{cases}$.

**Example 4.15**

Let $A=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix}$. Find, by hand, $AB$, $BA$, $(AB)^T$, $(BA)^T$, $A^T B^T$, $B^T A^T$, $A^{-1}$, $B^{-1}$, $(A^{-1})^{-1}$, $(AB)^{-1}$, $(BA)^{-1}$, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$, $(A^{-1})^T$, and $(A^T)^{-1}$. See what equals what.

**Definitions 4.1-4.5**

An $m\times n$ ***matrix*** is a rectangular array of numbers with $m$ rows and $n$ columns. $m$ and $n$ are called the ***dimensions*** of the matrix.

Matrices are most commonly denoted by capitol letters or single subscripted capitol letters. The numbers (***entries***) of a matrix are denoted by double subscripted lower case letters.

When a matrix is explicitly written out, it is delineated by square brackets or parentheses. Abstractly, you will see things like $A=\left[\,a_{ij}\,\right]$ where $a_{ij}$ represents the entry in the $i^{th}$ row and $j^{th}$ column. When the number of rows or columns goes above 9, we separate the $i$s and $j$s by commas; we won't be going there.

A matrix with only one row is called a ***row vector*** and a matrix with only one column is called a ***column vector***.

**Definitions 4.6 and 4.7**

***Matrix addition:*** Two matrices *with the exact same dimensions* can be added or subtracted thusly:

$$\left[a_{ij}\right] \pm \left[b_{ij}\right] = \left[a_{ij} \pm b_{ij}\right]$$

If either of the corresponding dimensions of two matrices is not the same, the matrices can neither be added nor subtracted.

***Scalar multiplication:*** A matrix can be multiplied by a ***scalar*** (number) thusly: $k\left[a_{ij}\right] = \left[k\,a_{ij}\right]$

**Definition 4.8**

The ***transpose*** of the $m \times n$ matrix $M$ is the $n \times m$ matrix, $M^{\mathbf{T}}$, that results from swapping the rows and columns of $M$.

**Definition 4.9: Generalized Matrix Multiplication**

If $A$ is an $m \times p$ matrix and $B$ is a $p \times n$ matrix, then the product $AB$ is the $m \times n$ matrix whose $ij^{\text{th}}$ entry is the dot product of the $i^{\text{th}}$ row of $A$ and the $j^{\text{th}}$ column of $B$

If $A$ is an $m \times p$ matrix and $B$ does not have $p$ rows, then the product $AB$ is not defined.

**Definitions 4.10-4.12 and Theorems 4.1-4.2**

The $n \times n$ matrix with entries of $1$ along the main diagonal and $0$ in every other slot is called the $n \times n$ ***identity matrix*** and is denoted as $I_n$.

If the products are defined, $AI_n = A$ and/or $I_n A = A$ for any compatible matrix $A$.

A **square** matrix $A$ with dimensions $n \times n$ is called ***invertible*** if there exists an $n \times n$ matrix $B$ with the property $AB = I_n$. If such a matrix exists we call it ***the inverse of A*** and symbolize it as $A^{-1}$.

If the $n \times n$ $A$ is invertible, then $AA^{-1} = A^{-1}A = I_n$.

We do not define inverse matrices for non-square matrices.

**Algorithm 4.1**

A game plan for determining, **by hand**, the inverse of the matrix $A=\begin{bmatrix}1&-1&0\\1&0&-1\\6&-2&-3\end{bmatrix}$ is shown below.

Let's execute the plan on the next page.

Let $A^{-1}=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}$. We need $AA^{-1}=I_3$ which gives us:

$$\begin{bmatrix}1&-1&0\\1&0&-1\\6&-2&-3\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}b_{11}-b_{21}&b_{12}-b_{22}&b_{13}-b_{23}\\b_{11}-b_{31}&b_{12}-b_{32}&b_{13}-b_{33}\\6b_{11}-2b_{21}-3b_{31}&6b_{12}-2b_{22}-3b_{32}&6b_{13}-2b_{23}-3b_{33}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

If we equate the columns of the respective matrices, we come up with three systems of three equations with three unknowns. Specifically:

$$\begin{cases}b_{11}-b_{21}=1\\b_{11}-b_{31}=0\\6b_{11}-2b_{21}-3b_{31}=0\end{cases},\quad\begin{cases}b_{12}-b_{22}=0\\b_{12}-b_{32}=1\\6b_{12}-2b_{22}-3b_{32}=0\end{cases},\text{ and }\begin{cases}b_{13}-b_{23}=0\\b_{13}-b_{33}=0\\6b_{13}-2b_{23}-3b_{33}=1\end{cases}$$

A not-so-close inspection should convince you that not only are the coefficient matrices for all three of the systems identical, but they are all in fact the matrix $A$! Since the row operations performed in Gaussian elimination are determined solely by the coefficient matrix, we may as well go ahead and solve all three systems simultaneously. The augmented matrix representation for these three systems is:

$$\left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\1&0&-1&0&1&0\\6&-2&-3&0&0&1\end{array}\right]$$

common coefficient matrix

Constant terms, from left to right, for the first, second, and third columns' related system of equations.

**Example 5.1**

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

**Example 5.2**

Use cofactors along the second row to find $\det(A)$ where $A=\begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

**Example 5.3**

Evaluate $\det(B)$ where $B=\begin{bmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{bmatrix}$.

**Example 5.4**

Use a determinant to find $\vec{u}\times\vec{v}$ where $\vec{u}=[1,7,-3]$ and $\vec{v}=[3,0,5]$.

**Example 5.5**: Let $B=\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$.

For each of the following matrices (each identified as $A$), describe the row operation that was affected upon $I_2$ to create $A$. Then find $\det(A)$ and compare its value to $\det(I_2)$. Next, find $AB$ and describe the difference between it and $B$. Finally, compare the values of $\det(AB)$ and $\det(B)$.

a. $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ b. $A=\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c.** $A=\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ d. $A=\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ e. $A=\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

**Example 5.6**

Determine $\det(A)$ and $\det(B)$ after first manipulating the matrices into upper triangular form.

$$A=\begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix} \qquad B=\begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{bmatrix}$$

**Example 5.7**

Illustrate Theorem 5.3 using $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

**Example 5.8**

Find the area of the parallelogram outline in Figure 1.

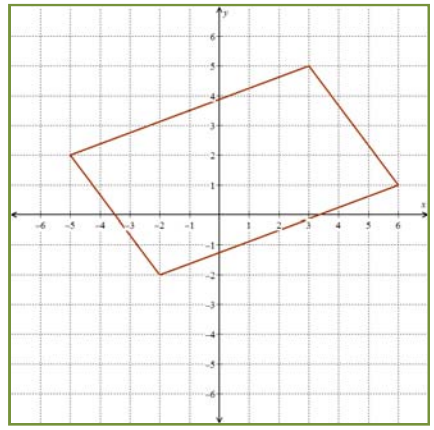


**Figure 1**: A parallelogram

**Example 5.9**

Use Cramer's Rule to find the solutions to each of the following matrix equations.

a. $\begin{bmatrix} 3 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -28 \\ -23 \\ 5 \end{bmatrix}$.

**Example 5.10**

Use the determinant and adjoint to find $A^{-1}$ where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

**Definition 5.1: Determinants (of square matrices)**

$$\det\left(\left[a_{11}\right]\right) = \left| a_{11} \right| = a_{11}$$

For a square matrix, $A$, with two or more rows we define the cofactor of entry ***a<u>ij</u>***, $C_{ij}$, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the $i^{th}$ row and $j^{th}$ column from $A$. Then using any row of $A$ or any column of $A$:

$$\det(\mathrm{A}) = \sum_{j=1}^{n}\left[a_{ij}\, C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}\, C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed $i^{th}$ row of $\mathrm{A}$ whereas in the second formula we are summing along a fixed $j^{th}$ column of $A$.

**Definition 5.2: Elementary Matrices**

An ***<u>elementary matrix</u>*** is a matrix that can be created from an identity matrix via one elementary row operation.

**Theorem 5.1: Elementary Row Operations and Determinants**

Suppose that $A$ and $B$ are square matrices of equal dimension; suppose further that $B$ can be created from $A$ via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of $A$ to a different row of $A$, then $\det(B) = \det(A)$
- if the operation is swapping two rows of $A$, then $\det(B) = -\det(A)$
- if the operation is multiplying a row of $A$ by the real number $k$, then $\det(B) = k \cdot \det(A)$.

**Definitions 5.3-5.4 and Theorem 5.2**

An ***<u>upper triangular matrix</u>*** is a matrix where every entry *<u>below</u>* the main diagonal is zero.

A ***<u>lower triangular matrix</u>*** is a matrix where every entry *<u>above</u>* the main diagonal is zero.

The determinant of any $n \times n$ triangular matrix $B$ is given by the formula $\det(B) = \prod_{i=1}^{n} b_{ii}$ .

**Theorems 5.3 and 5.4**

- If $A$ and $B$ are like-sized square matrices, then $\det(AB) = \det(A)\det(B)$.
- If $A$ is any square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

**Theorem 5.5: A Little Geometry**

$\left| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right|$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

**Theorem 5.6: Cramer's Rule**

Consider the matrix equation $A\vec{x} = \vec{b}$ where A is a nonsingular square matrix. Define $A_i(\vec{b})$ to be the matrix derived from A by replacing the ith colum of A with $\vec{b}$. Then the solution to $A\vec{x} = \vec{b}$ can be found using the formula $x_i = \dfrac{\det\left(A_i\left(\vec{b}\right)\right)}{\det(A)}$ for each $x_i$.

**Theorem 5.7: An algorithm for finding inverse matrices**

The matrix of cofactors of a square matrix *A* is the matrix that results from replacing each of its entries by their corresponding cofactors.

***The Adjoint (Adjugate) of A*** is the transpose of *A*'s matrix of cofactors.

The inverse of a nonsingular square matrix *A* is $A^{-1} = \dfrac{1}{\det(A)}\mathrm{Adj}(A)$.

Please note that this implies that the matrix *A* is nonsingular if and only if $\det(A) \neq 0$.This also implies, albeit less directly, that the square system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.