Example 1.1

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases} \quad -3E_1 + E_2 \to E_2 \begin{cases} 2x_1 - 3x_2 = 8 \\ 10x_2 = -60 \end{cases}$$

$$\tfrac{1}{10}E_2 \to E_2 \begin{cases} 2x_1 - 3x_2 = 8 \\ x_2 = -6 \end{cases}$$

$$3E_2 + E_1 \to E_1 \begin{cases} 2x_1 = -10 \\ x_2 = -6 \end{cases}$$

$$\tfrac{1}{2}E_1 \to E_1 \begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$$

$E_1 : 2(-5) - 3(-6) = 8$ ✓

$E_2 : 6(-5) + (-6) = -36$ ✓

The solution is $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$

Let's do this again efficiently!

Augmented Matrix

$$\begin{array}{ccc} & & \\ \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right] \\ \uparrow \quad \uparrow \quad \uparrow \\ x_1 \quad x_2 \quad \text{constants} \end{array} \quad -3R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 10 & -60 \end{array}\right]$$

$$\tfrac{1}{10}R_2 \to R_2 \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 1 & -6 \end{array}\right]$$

$$3R_2 + R_1 \to R_1 \left[\begin{array}{cc|c} 2 & 0 & -10 \\ 0 & 1 & -6 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -6 \end{array}\right]$$

The solution is $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$ checked

Example 1.2

a)

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right] R_1 \Leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ -5 & 1 & -2 & -7 \\ 2 & -5 & -3 & -23 \end{array}\right]$$

$$\begin{array}{l} 5R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & -11 & -5 & -29 \end{array}\right]$$

$$\tfrac{11}{16}R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & -47/16 & -47/2 \end{array}\right]$$

We have a system equivalent to the original system

$$\begin{cases} x_1 + 3x_2 + x_3 = 3 \\ 16x_2 + 3x_3 = 8 \\ -\frac{47}{16}x_3 = -\frac{47}{2} \end{cases}$$

Solve, bottom up, back subbing along the way

$-\frac{16}{47}E_3$: $x_3 = 8$

Back sub to E_2 : $16x_2 + 3(8) = 8 \Rightarrow x_2 = -1$

Backsub to E_1 : $x_1 + 3(-1) + (8) = 3 \Rightarrow x_1 = -2$

Check

$2(-2) - 5(-1) - 3(8) = -23$ ✓

$-5(-2) + (-1) - 2(8) = -7$ ✓

$(-2) + 3(-1) + (8) = 3$ ✓

The solution is: $\begin{cases} x_1 = -2 \\ x_2 = -1 \\ x_3 = 8 \end{cases}$

b. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ -2 & 3 & -2 & 2 & -3 \\ 0 & -1 & -2 & 0 & -11 \\ 5 & -10 & 0 & -3 & -1 \end{array}\right]$

$2R_1 + R_2 \to R_2$
$-5R_1 + R_4 \to R_4$
$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$

$-R_2 + R_3 \to R_3$
$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$

$-2R_3 + R_4 \to R_4$
$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$

The 4th Eq. of our equivalent system is $0 = 11$. No way! This is a <u>contradiction</u>.

The solution set to the given system is ϕ.

a. The pivot columns of A are C1 and C3
Col 1 Col 3

$\left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

pivot pivot

no contradictions
$[\underbrace{3x_3 = 0}_{\text{not } 3=0} \Rightarrow x_3 = 0]$

fewer non-zero rows than variables.

$\therefore$ The system has an unlimited number of solutions (Bullet 3)

- $n = 3$ (# of variables) # of PC < n
  # of pivot columns = 2
  and constant column is not a pivot column,

b. $B = \begin{bmatrix} 1 & 0 & \vdots & 7 \\ 0 & 8 & \vdots & 6 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$ # of variables : $n = 2$

pivot columns: Col 1, col 2

n = # of pivot columns

$\therefore$ The implied system has exactly one solution.

c. $C = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

leading entries: $C_{11} = -2$, $C_{22} = 4$, $C_{35} = 6$

Pivot columns: Col 1, col 2, col 5 (rightmost column)

Whenever the rightmost column is a pivot column, the system has no solutions. (In this case, the third row translates to $0 = 6$). Note that the system is inconsistent.

1.4 : a)

$\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 6 & -6 & -1 \end{array}\right]$ $-3R_1 + R_2 \to R_2$ $\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & -15 & 5 \end{array}\right]$

$\frac{1}{5}R_2 + R_1 \to R_1$ $\left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & -15 & 5 \end{array}\right]$

$\frac{1}{2}R_1 \to R_1$

$-\frac{1}{15}R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & -1/3 \end{array}\right]$

The only solution is $\begin{cases} x_1 = -1/2 \\ x_2 = -1/3 \end{cases}$

b. $\left[\begin{array}{ccc|c} 0 & 2 & -6 & -2 \\ 4 & -1 & 3 & 1 \\ -1 & 3 & -8 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 4 & -1 & 3 & 1 \\ 0 & 2 & -6 & -2 \end{array}\right]$

$$\begin{array}{c} 4R_1 \to R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 11 & -29 & -15 \end{array}\right]$$

$$-11R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

$$\begin{array}{c} 2R_3 + R_1 \to R_1 \\ \frac{3}{4}R_3 + R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|c} -1 & 3 & 0 & -12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

$$\begin{array}{c} -3R_2 + R_1 \to R_1 \\ \frac{1}{4}R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$-1 \cdot R_1 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $\begin{cases} x_1 = 0 \\ x_2 = -4 \\ x_3 = -1 \end{cases}$.

c) $$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 3 & -2 & 1 & 5 & -1 \\ 2 & 4 & -1 & 5 & 8 \\ -3 & -1 & 1 & -7 & -6 \end{array}\right]$$

$3R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$-3R_1 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & -19 & 7 & -7 & -33 \end{array}\right]$$

$-R_2 + R_3 \to R_3$

$\frac{19}{16} R_2 + R_4 \to R_4$

$-R_1 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & -6 & 2 & 0 & -9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{17}{16} & -\frac{17}{16} & -\frac{34}{16} \end{array}\right]$$

$\frac{16}{17} R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & -6 & 2 & 0 & -9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & -6 & 2 & 0 & -9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-2R_3 + R_1 \to R_1$

$5R_3 + R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 1 & -6 & 0 & 2 & -5 \\ 0 & 16 & 0 & 0 & 16 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{16}R_2 \to R_2$
$$\left[\begin{array}{cccc|c} 1 & -6 & 0 & 2 & -5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$6R_2 + R_1 \to R_1$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A system equivalent to the original system is:

$$\begin{cases} x_1 \qquad\qquad + 2x_4 = 1 \\ \quad x_2 \qquad\qquad\quad = 1 \\ \qquad x_3 - x_4 = -2 \\ \qquad\qquad\quad 0 = 0 \end{cases}$$

Solve each equation for the left most variable.

$$\begin{cases} x_1 = -2x_4 + 1 \\ x_2 = 1 \\ x_3 = x_4 - 2 \end{cases}$$

any variable on the right side of "=" is called a free variable.

In this problem, x_1 & x_3 is a dependendent variable (dependendent upon the value assigned to x_4). In this problem x_2 is a fixed variable (value is fixed at 1).

Check

$$-x_1 + 6x_2 - 2x_3 = -(-2x_4+1) + 6(1) - 2(x_4-2)$$
$$= 2x_4 - 1 + 6 - 2x_4 + 4$$
$$= 9 \checkmark$$

$$3x_1 - 2x_2 + x_3 + 5x_4 = 3(-2x_4+1) - 2(1) + (x_4-2) + 5x_4$$
$$= -6x_4 + 3 - 2 + x_4 - 2 + 5x_4$$
$$= -1 \checkmark$$

The <u>general</u> solution to the system of equations is

$$\begin{cases} x_1 = -2x_4 + 1 \\ x_2 = 1 \\ x_3 = x_4 - 2 \\ x_4 \text{ is free} \end{cases}$$

<u>One</u> specific solution (arbitrarily picking 3 for the value of x_4) is:

$$\begin{cases} x_1 = -5 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 3 \end{cases}$$

1.5) $\left[\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right]$

A system equivalent to the given system is:

$$\begin{cases} x_1 \quad\quad + x_4 = -4 \\ x_2 - x_3 + 2x_4 = -3 \end{cases}$$

$\therefore$ The eneral solution is

$$\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

Two specific solutions are

$$\begin{cases} x_1 = -9 \\ x_2 = -10 \\ x_3 = 3 \\ x_4 = 5 \end{cases} \qquad \begin{cases} x_1 = -8 \\ x_2 = -5 \\ x_3 = 6 \\ x_4 = 4 \end{cases}$$

b. $\begin{bmatrix} 2 & 6 & 5 & \vdots & -2 \\ -1 & -3 & 3 & \vdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$

A system equivalent to the given system is

$$\begin{cases} x_1 + 3x_2 \quad = -1 \\ \qquad x_3 = 0 \end{cases}$$

The general Solution is $\begin{cases} x_1 = -3x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$

Two specific solutions are:

$$\begin{cases} x_1 = -4 \\ x_2 = 1 \\ x_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} x_1 = 10 \\ x_2 = -3 \\ x_3 = 0 \end{cases}$$

Check

$2(-4) + 6(1) + 5(0) = -2$ ✓

$-(-4) - 3(1) + 3(0) = 1$ ✓

Let x_1 be the number of Ethane (C_2H_6) molecules added to the mixture.

Let x_2 be the number of Oxygen (O_2) molecules added to the mixture.

Let x_3 be the number of resultant CO_2 molecules

Let x_4 be the number of resutant H_2O molecules.

$$x_1 C_2H_6 + x_2 O_2 \rightarrow x_3 CO_2 + x_4 H_2O$$

Balancing Carbon atoms: $2x_1 = x_3$

Balance Hydrogen atoms: $6x_1 = 2x_4$

Balance Oxygen atoms: $2x_2 = 2x_3 + x_4$

Our System is:

$$\begin{cases} 2x_1 \quad - x_3 \quad = 0 \\ 6x_1 \quad - 2x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -7/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{array}\right]$$

By inspection, the general solution is

$$\begin{cases} x_1 = \frac{1}{3} x_4 \\ x_2 = \frac{7}{6} x_4 \\ x_3 = \frac{2}{3} x_4 \\ x_4 \text{ is free} \end{cases}$$

Seeing as we can only have a whole number of each atom type,

I choose $x_4 = 6$.

$\therefore$ A balanced equation is

$$2C_2H_6 + 7O_2 \rightarrow 4CO_2 + 6H_2O$$

Table 1: Flow rates

node	rate in = rate out
A	$25+50 = x_1 + x_4$
B	$x_1 = x_2 + 50$
C	$x_2 + x_3 + 25 = 150$
D	$100 + x_4 = x_3$

our flow system of equations is:

$$\begin{cases} x_1 + x_4 = 75 \\ x_1 - x_2 = 50 \\ x_2 + x_3 = 125 \\ x_3 - x_4 = 100 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 1 & -1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 125 \\ 0 & 0 & 1 & -1 & 100 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

our general solution is:

$$\begin{cases} x_1 = -x_4 + 75 \\ x_2 = -x_4 + 25 \\ x_3 = x_4 + 100 \\ x_4 \text{ is free} \end{cases}$$

Since all flow rates must be non-negative, $0 \le x_4 \le 25$

$\therefore$ The extreme flow rates are

$$50 \le x_1 \le 75$$
$$0 \le x_2 \le 25$$
$$100 \le x_3 \le 125$$
$$0 \le x_4 \le 25$$

1.6

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right]$$

To determine the nature of the solution, you only need to reduce to REF. The advantage of RREF is that you can easilly identify the actual solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right] \xrightarrow[2R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & -6 & 9 & 2a+c \end{array}\right]$$

$$\xrightarrow{3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & 0 & 0 & -a+3b+c \end{array}\right]$$

$\therefore$ The system has a solution iff $-a+3b+c=0$. So there's a solution iff $c = a - 3b$

(Sort of) check

True

$a=1, b=1, c=-2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & -1 & 1 \\ -2 & -8 & 5 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 1 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General Solution: $\begin{cases} x_1 = -\frac{7}{2}x_3 + 1 \\ x_2 = \frac{3}{2}x_3 \\ x_3 \text{ is free} \end{cases}$

False not -2

$a=1, b=1, c=4$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & -1 & 1 \\ -2 & -8 & 5 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$0 \ne 1$

no solutions