Example 1.1
Use an augmented matrix to mimic the elimination method for solving the following linear system of equations.

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}.$$

Example 1.2
Use the method of Gaussian elimination to find an echelon form of the augmented matrix representation for each of the following systems of equations and use that matrix to determine the solution to the system of equations.

a. $$\begin{cases} 2x_1 - 5x_2 - 3x_3 = -23 \\ -5x_1 + x_2 - 2x_3 = -7 \\ x_1 + 3x_2 + x_3 = 3 \end{cases}$$

b. $$\begin{cases} x_1 - 2x_2 + x_4 = 4 \\ -2x_1 + 3x_2 - 2x_3 + 2x_4 = -3 \\ -x_2 - 2x_3 = -11 \\ 5x_1 - 10x_2 - 3x_4 = -1 \end{cases}$$

Example 1.3

Several augmented row echelon form matrices are given below (and on the next page). For each matrix, identify the pivot columns and state the nature of the solution set for the associated system of equations.

a. $$\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

b. $$\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$$

c. $$\mathbf{C} = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example 1.4

Solve the following systems of equations using the Gauss-Jordan elimination method.

a. $$\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$$

b. $$\begin{cases} 2x_2 - 6x_3 = -2 \\ 4x_1 - x_2 + 3x_3 = 1 \\ -x_1 + 3x_2 - 8x_3 = -4 \end{cases}$$

c. $$\begin{cases} -x_1 + 6x_2 - 2x_3 = 9 \\ 3x_1 - 2x_2 + x_3 + 5x_4 = -1 \\ 2x_1 + 4x_2 - x_3 + 5x_4 = 8 \\ -3x_1 - x_2 + x_3 - 7x_4 = -6 \end{cases}$$

Example 1.5

State a ***general solution*** to the following systems of equations as well as two ***specific solutions*** to the system.

a. $\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ \quad\quad x_2 - x_3 + 2x_4 = -3 \end{cases}$ b. $\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$

Example 1.6

Under what conditions is the following system of equations consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$$

Example 1.7: Application: Balancing Chemical Equations

Ethane and Oxygen combine to produce Carbon Dioxide and steam. Formally, this is represented by the equation $\mathrm{C_2H_6 + O_2 \rightarrow CO_2 + H_2O}$. Let's put our newfound skills to use and balance this equation. Speaking of putting things to use … let's use our calculator to find the RREF form of the matrix.

Example 1.8: Application: Network Analysis

A network is most easily thought of as a city street system. The intersections are technically called ***nodes*** or ***junctions*** and each directed stretch of road between intersections is called a ***branch***. Because branches are directed, if there is a two-way street between two intersections the corresponding network will have two branches between the corresponding nodes. We assign values or variables to each branch; those values and variables could conceptually represent flow-rates or flow-amounts along those branches. In order for the network to be valid, **the total flow into the network must equal the total flow out of the network**. The values and variables in Figure 1 represent traffic flow rates (vehicles/quarter-hour) in a small section of a city street system. Let's determine the minimum and maximum flow rates through each of the variable branches.

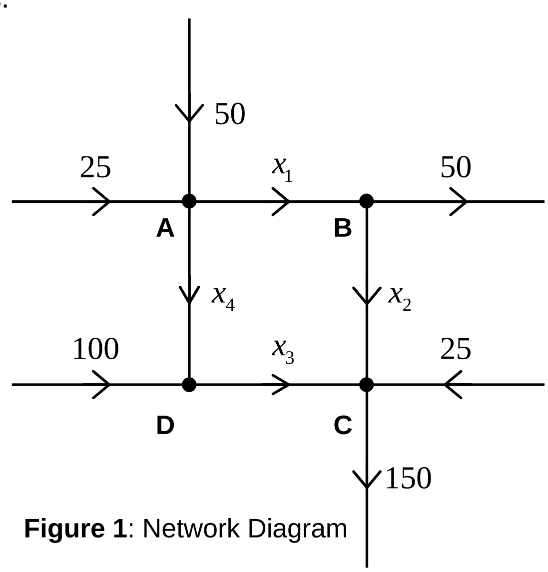

Figure 1: Network Diagram

Definitions 1.1-1.5

The Gaussian elimination process for solving systems of linear equations is predicated upon the three ***elementary row operations***. The three elementary row operations are:

1. Interchanging two rows of the matrix
2. Replacing one row of the matrix with a non-zero multiple of itself
3. Replacing one row of a matrix with itself added to a multiple of another row of the matrix

Two matrices are said to be ***row equivalent*** if one matrix can be transformed into the other via a series of elementary row operations. The first non-zero entry of any row of a matrix is called ***the leading entry*** of that row. A matrix is said to be in **(*row*) *echelon form*** if the matrix satisfies both of the following properties.

i. Every row that contains nothing but zeros occurs at the bottom of the matrix.
ii. The leading entry of any non-zero row appears to the right of the leading entry in the row directly above it.

The process of transforming a matrix into a row equivalent matrix of echelon form is called the ***Gaussian elimination process***.

Definitions 1.6-1.9 and Theorem 1.1

A ***pivot position*** in a matrix is an entry position that corresponds to a leading entry in a row echelon form of the matrix. Any column that contains a pivot entry is called a ***pivot column***.

If A is an augmented matrix representing a system of equation with n variables, then:

- the system of equations has exactly one solution if and only if A has exactly n pivot columns all of which lie to the left of the augment line;
- the system of equations has no solutions if and only if the right-most column of A is a pivot column;
- the system of equations has an unlimited number of solutions if and only if there are less than n pivot columns in A and the right-most column of A is not a pivot column.

A system of equations with at least one solution is called ***consistent***. A system of equations with no solutions is called ***inconsistent***.

Gauss-Jordan Elimination Method

The ***Gauss-Jordan elimination method*** follows the same row manipulation rules as Gaussian elimination. What distinguishes the two methods is that the Gaussian method stops once you've reduced the matrix to row echelon form where-as the Gauss-Jordanian method requires you to further reduce the matrix to ***reduced (row) echelon form***. That is, the simplified matrix must meet each of the following properties:

i. Every row that contains nothing but zeros occurs at the bottom of the matrix.
ii. The leading entry of any non-zero row appears to the right of the leading entry in the row directly above it.
iii. All entries directly above or below a leading entry are zero.
iv. Every leading entry is 1.

When manipulating a matrix into reduced row echelon form, there are essentially three tasks that need to be completed. In the list below, you always want to complete task A first. Tasks B and C can be done in either order.

A. Manipulate every entry below a leading entry to zero. This task should be worked left-to-right; i.e., first create the necessary zeros in the first column, then the second column, then the third column, etc. Remember that the leading entries need to move rightward as you move down the rows of the matrix. Occasionally you will need to swap rows to maintain this cascading effect. **Always use the row containing the leading entry to create the zeros below that leading entry**
B. Manipulate every entry above a leading entry to zero. This task should be worked right-to-left. **Always use the row containing the leading entry to create the zeros above that leading entry.**
C. Multiply each of the rows by the necessary constants so that each leading entry is 1.

Definitions 1.10-1.13

Suppose that A is the augmented representation of a ***consistent system*** with *n* variables that has ***an unlimited number of solutions***.

The ***free variables*** in the ***general solution set*** correspond to the non-pivot columns of A that lie to the left of the augment line. The other variables in the system are called ***basic variables***. The value of any given basic variable might be fixed or it might be dependent upon the value(s) assigned to one or more of the free variables.

Specific solutions to the system are determined by assigning specific values to the free variables and then determining the associated values of the basic variables.