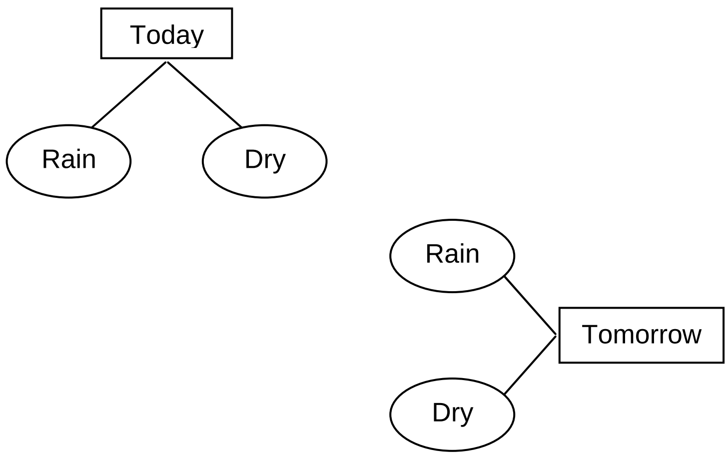
Example 11.1

It rains a lot in Gloomtown. If it rains today, there is a 90% probability that it will rain tomorrow. If it is dry today, there is an 80% chance it will be dry tomorrow. What percentage of days does it rain in Gloomtown?

a. Let's start by coming up with a transition matrix for this Markov Chain.



b. Let's familiarize ourselves with the process. What does a state vector represent in the context of this problem?

c. Let's establish that if it rains today, then the state vector for tomorrow is $P^1[1,0]^{\mathrm{T}}$, the state vector two days from now is $P^2[1,0]^{\mathrm{T}}$ and the state vector 3 days from now is $P^3[1,0]^{\mathrm{T}}$

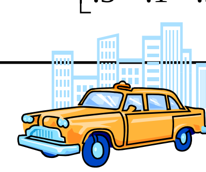
d. If it rains today, what are the state vectors for each of the next three days? What about if it's dry today? Then project into the future … if it rains today, what are the state vectors 50 days from now, 51 days from now, and 52 days from now? What about if it's dry today? What is the *steady state* vector for this Markov Chain? What does it tell us in the context of this problem?

e. Let's use eigenvalues to determine the steady state vector for Gloomtown's rainy situation.

Example 11.2

Pretend that Manhattan only has Midtown, the Upper East Side, and the Upper West Side. Suppose that cabbie shifts change only at 6 am and 6 pm. Suppose that a state vector for the distribution of cabs has form $[\text{Midtown}, \text{UES}, \text{UWS}]^{\mathrm{T}}$ and that the transition matrix over each 5 minute interval between 6 am and 6 pm is the matrix P. Find the steady state vector for this model. Suppose that the cabs are evenly distributed at the start of the 6 am shift; how well does the steady state vector predict the distribution of the cabs at the end of that shift?

$$P = \begin{bmatrix} .5 & .3 & .4 \\ .2 & .6 & .1 \\ .3 & .1 & .5 \end{bmatrix}$$



Example 11.3

Explain ***geometrically*** why the rotation matrix $R=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$ cannot possibly have any real number eigenvalues for $0<\theta<\pi$.

Example 11.4

What are bases for the kernel and range of the linear transformation $T(\vec{x})=A\vec{x}$ where $A=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$. What does this imply must be true about A^2.

Definitions 11.1-11.4: Stochastic Vectors, Stochastic Matrices, and Markov Chains

A ***stochastic vector*** is vector containing no negative components and whose components sum to 1.

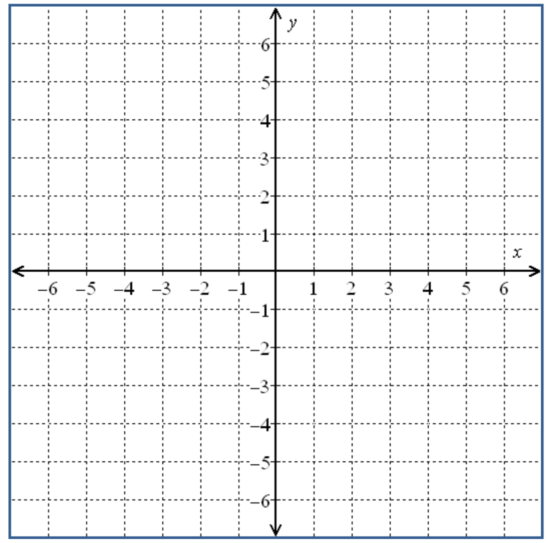
A ***stochastic matrix*** is a matrix whose column vectors are stochastic vectors.

Suppose that any given member of a set must be in exactly 1 of a finite number of states at any given time. Suppose further that the state of each member is noted over regular time intervals. Then the stochastic vector $\vec{v}_k=\left[q_1^{(k)},q_2^{(k)},\ldots,q_n^{(k)}\right]$ where $q_i^{(k)}$ is the probability that a given member of the set is in state i at time k is called a ***state vector***

Suppose that p_{ij} is the probability that a member in state j will transition to state i the next time an account is made. Suppose further that these probabilities do not change over time. Then the sequence $\vec{v}_1,\vec{v}_2,\vec{v}_3,\ldots$ is called a ***Markov Chain***.

Furthermore, $\vec{v}_{k+1}=P\vec{v}_k$ where $P=\left[p_{ij}\right]_{n\times n}$ and P is called the ***transition matrix*** for the Markov Chain.

Example 12.1
Determine and illustrate the projection vector, component vector, and scalar component when $\vec{y}=[-4,2]^{\mathrm{T}}$ and $\vec{u}=[5,1]^{\mathrm{T}}$.



Example 12.2
Let $\beta=\left\{[1,1,1]^{\mathrm{T}},[-1,-1,2]^{\mathrm{T}},\ [3,-3,0]^{\mathrm{T}}\right\}$. Show that β is orthogonal and, therefore, forms a basis for $\mathbb{R}^3$. Then find $[\vec{y}]_\beta$ where $\vec{y}=[5,-2,0]^{\mathrm{T}}$.

Example 12.3
Let's find an orthonormal basis for the column space of $A=\begin{bmatrix}-2 & 1 & 5 & 1\\ 4 & -2 & -10 & 1\\ 3 & 0 & -6 & 1\\ -1 & 5 & 7 & 0\end{bmatrix}$ and find the coordinates of $\vec{y}=[5,-10,-6,7]^{\mathrm{T}}$ in terms of that basis.

Example12. 4
Prove that if W is a subspace of $\mathbb{R}^n$, then $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Example 12.5
Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Illustrate that $(\mathrm{row}(A))^\perp=\mathrm{nul}(A)$.

Example 12.6
Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Illustrate that $(\mathrm{col}(A))^\perp=\mathrm{nul}(A^{\mathrm{T}})$.

Example 12.7
Let $A=\begin{bmatrix}1 & 2 & 1 & 4\\ 2 & 1 & 5 & -2\\ -2 & 2 & -8 & 0\\ 3 & 2 & 7 & 10\end{bmatrix}$ and note that $A\sim\begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$. Find a basis for each of the following vector spaces ***<u>without actually transposing the matrix A</u>***. In each case, write a few words so that the rationale for whatever action/conclusion you take/make is clear.

a. $\mathrm{null}(A^{\mathrm{T}})^\perp$

b. $\mathrm{col}(A^{\mathrm{T}})^\perp$

Example 12.8

Let $\vec{u}_1 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\vec{u}_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$, and $\vec{u}_3 = \begin{bmatrix} -1 & 1 & 1 & 3 \end{bmatrix}^{\mathrm{T}}$. Let $W = \operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\vec{y} = \begin{bmatrix} 2 & 1 & 3 & 4 \end{bmatrix}^{\mathrm{T}}$. Express $\vec{y}$ as the sum of two vectors, one from W and the other from $W^{\perp}$.

Example 12.9

A basis from $\mathbb{R}^3$ for the $x_1\,x_2 - \text{plane}$ is $\{\vec{u}_1, \vec{u}_2\}$ where $\vec{u}_1 = \begin{bmatrix} 3 & -4 & 0 \end{bmatrix}^{\mathrm{T}}$ and $\vec{u}_2 = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}^{\mathrm{T}}$. Use this basis to determine the point on the $x_1\,x_2 - \text{plane}$ that is closest to the point $(2,7,5)$.

Example 12.10

Let $\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find an orthonormal basis for $\operatorname{span}\{\vec{x}_1, \vec{x}_2\}$. Next, express $\vec{x}_1$ and $\vec{x}_2$ in terms of that new basis.

Definitions 12.1 – 12.3: Projection vectors in $\mathbb{R}^n$

The ***orthogonal projection of*** $\vec{y}$ ***onto*** $\vec{u}$ is the vector $\operatorname{proj}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The ***component of*** $\vec{y}$ ***orthogonal to*** $\vec{u}$ is $\vec{y} - \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

The quantity $\operatorname{comp}_{\vec{u}}(\vec{y}) = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ is called the ***scalar component of*** $\vec{y}$ ***onto*** $\vec{u}$.

Definition 12.4 and Theorem 12.1

A set of non-zero vectors, $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$, is called an ***orthogonal set*** if and only if each pair of vectors in the set is orthogonal; that is, it is an orthogonal set if and only if $\vec{u}_i \cdot \vec{u}_j = 0 \;\forall\; i \neq j$.

The vectors from any orthogonal set are linear independent. Consequently any orthogonal set of n vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Theorem 12.2: Change of coordinates in $\mathbb{R}^n$

Let $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal set from $\mathbb{R}^n$. Then every vector $\vec{y}$ in $\mathbb{R}^n$ can be expressed as $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ where $c_i = \operatorname{comp}_{\vec{u}_i}(\vec{y})$.

That is, $[\vec{y}]_{\beta} = [c_1, c_2, \ldots, c_n]^{\mathrm{T}}$ where $c_i = \operatorname{comp}_{\vec{u}_i}(\vec{y})$.

Theorem *12.3:* *The Gramm-Schmidt process* for finding an orthogonal basis

Let $\beta = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p \}$ be a linearly independent set of vectors.

Then the set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \}$ is an orthogonal basis for the span of β where:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \operatorname{proj}_{\vec{v}_1}(\vec{x}_2) \\ \vec{v}_3 &= \vec{x}_3 - \operatorname{proj}_{\vec{v}_1}(\vec{x}_3) - \operatorname{proj}_{\vec{v}_2}(\vec{x}_3) \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \operatorname{proj}_{\vec{v}_1}(\vec{x}_p) - \operatorname{proj}_{\vec{v}_2}(\vec{x}_p) - \cdots - \operatorname{proj}_{\vec{v}_{p-1}}(\vec{x}_p) \end{aligned}$$

Furthermore, the set $\gamma = \left\{ \dfrac{\vec{v}_1}{|\vec{v}_1|}, \dfrac{\vec{v}_2}{|\vec{v}_2|}, \ldots, \dfrac{\vec{v}_p}{|\vec{v}_p|} \right\}$ is an ***orthonormal basis*** for the span of β.

Definition 12.5 and Theorems 12.4 – 12.6

If W is a subspace of $\mathbb{R}^n$, then the ***orthogonal compliment*** of W is the set of vectors, $W^{\perp}$, that are orthogonal to every vector in W.

- $W^{\perp}$ is a subspace of $\mathbb{R}^n$.
- $\left(\operatorname{row}(A)\right)^{\perp} = \operatorname{nul}(A)$.
- $\left(\operatorname{col}(A)\right)^{\perp} = \operatorname{nul}\left(A^{\mathrm{T}}\right)$.

Theorem 12.7: The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Then there exists *unique* vectors $\vec{w}$ and $\vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$, such that $\vec{y} = \vec{w} + \vec{z}$.

In fact, if the set $\{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \}$ is an orthogonal basis for W, then

$$\vec{w} = \operatorname{proj}_{\vec{u}_1}(\vec{y}) + \operatorname{proj}_{\vec{u}_2}(\vec{y}) + \cdots + \operatorname{proj}_{\vec{u}_p}(\vec{y}) \text{ and } \vec{z} = \vec{y} - \vec{w}$$

Theorem 12.8

The $m \times n$ matrix U has orthonormal columns if and only if $U^{\mathrm{T}}U = I_n$.

Example 13.1

Solve the system $\dfrac{d\mathbf{y}}{dt} = A\,\mathbf{y}$ given that $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

Example 13.2

Solve the system $\begin{cases} y_1' &= y_1 + 2y_2 \\ y_2' &= -y_1 + 4y_2 \end{cases}$ given that $y_1(0) = -5$ and $y_2(0) = 0$. Begin by writing the system in the form $\mathbf{y}' = A\,\mathbf{y}$ and "uncoupling" the system by diagonalizing A and making substitutions based upon $\mathbf{y} = P\,\mathbf{w}$ and $\mathbf{y}' = P\,\mathbf{w}'$.

Example 13.3

Solve the system $\begin{cases} y_1' &= y_1 + 2y_2 \\ y_2' &= 2y_1 + 4y_2 \end{cases}$ given that $y_1(0) = 1$ and $y_2(0) = -1$. Begin by writing the system in the form $\mathbf{y}' = A\,\mathbf{y}$ and "uncoupling" the system by diagonalizing A and making substitutions based upon $\mathbf{y} = P\,\mathbf{w}$ and $\mathbf{y}' = P\,\mathbf{w}'$.

Example 13.4

Solve the system $\begin{cases} y_1' &= y_2 \\ y_2' &= -y_1 \end{cases}$ given that $y_1(0) = 3$ and $y_2(0) = 3$. Begin by writing the system in the form $\mathbf{y}' = A\,\mathbf{y}$ and "uncoupling" the system by diagonalizing A and making substitutions based upon $\mathbf{y} = P\,\mathbf{w}$ and $\mathbf{y}' = P\,\mathbf{w}'$.