Example 9.1

Consider the $\mathbb{R}^2$ basis $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$. Answer each of the following questions relative to β.

a. Determine $\vec{x}$ if $[\vec{x}]_\beta = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

b. Determine $[\vec{x}]_\beta$ if $\vec{x} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$.

a. $$\vec{x} = (-4)\begin{bmatrix} 3 \\ -1 \end{bmatrix} + (7)\begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -26 \\ 39 \end{bmatrix}$$

b. we need to determine x_1, x_2 such that

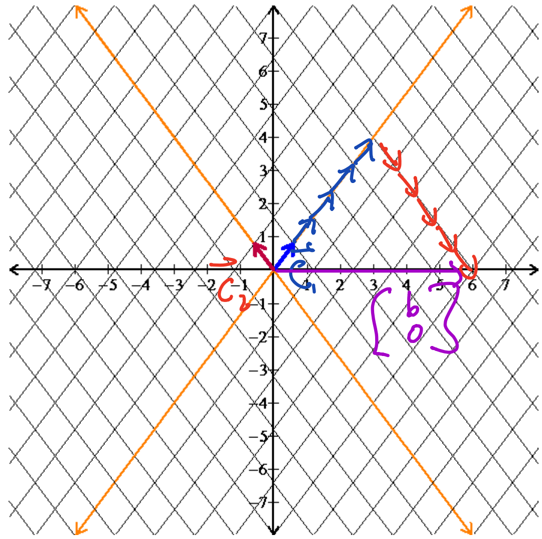
$$x_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -2 & 26 \\ -1 & 5 & -39 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array}\right]$$

$$\therefore \quad [\vec{x}]_\beta = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

Example 9.2

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{ \vec{c}_1, \vec{c}_2 \}$. Find the change-of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_\gamma$.



Let's begin by finding $[\vec{e}_1]_\gamma$ and $[\vec{e}_2]_\gamma$

i.e., let's solve

$$x_1 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} + x_2 \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

& $$y_1 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} + y_2 \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 3/5 & -3/5 & 1 & 0 \\ 4/5 & 4/5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & 5/8 \\ 0 & 1 & -5/6 & 5/8 \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\tfrac{5}{6}\right)\vec{c}_1 + \left(-\tfrac{5}{6}\right)\vec{c}_2 \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\tfrac{5}{8}\right)\vec{c}_1 + \left(\tfrac{5}{8}\right)\vec{c}_2$$

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= a\left(\tfrac{5}{6}\vec{c}_1 + \left(-\tfrac{5}{6}\right)\vec{c}_2\right) + b\left(\left(\tfrac{5}{8}\right)\vec{c}_1 + \left(\tfrac{5}{8}\right)\vec{c}_2\right)$$

$$= \left(\tfrac{5}{6}a + \tfrac{5}{8}b\right)\vec{c}_1 + \left(-\tfrac{5}{6}a + \tfrac{5}{8}b\right)\vec{c}_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_\gamma = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix}_\gamma = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

← Change of basis matrix

$\underset{\gamma \leftarrow \mathcal{B}}{P}$

Example 9.3

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

$$\left[\begin{array}{c|c} \text{"new"} & \text{"old"} \\ \text{basis} & \text{basis} \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -2 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1 & 0 \end{array} \right]$$

$$\therefore \underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix}$$

$$\therefore [\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$$

$$= \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -9 \\ 0 \end{bmatrix}$$

Check

using β-coordinates

$$\vec{x} = 3\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-4)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$$

using γ-coordinates

$$\vec{x} = 8\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-9)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$$

$\checkmark$

Explanation of the mechanics involved in Example 9.3

$$\begin{array}{cccccc} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 & \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1 & 0 \end{array}\right]$$

The row equivalency above gives us:

$$\vec{b}_1 = \tfrac{8}{3}\vec{g}_1 - \tfrac{5}{3}\vec{g}_2 - \tfrac{2}{3}\vec{g}_3, \quad \vec{b}_2 = 0\vec{g}_1 + 0\vec{g}_2 + 1\vec{g}_3, \quad \vec{b}_3 = 0\vec{g}_1 + 1\vec{g}_2 + 0\vec{g}_3.$$

Now suppose that $[\vec{x}]_\beta = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, i.e. $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3$.

Substituting from above we have:

$$\vec{x} = x_1\left(\tfrac{8}{3}\vec{g}_1 - \tfrac{5}{3}\vec{g}_2 - \tfrac{2}{3}\vec{g}_3\right) + x_2(\vec{g}_3) + x_3(\vec{g}_2)$$

$$= \tfrac{8}{3}x_1\vec{g}_1 + \left(-\tfrac{5}{3}x_1 + x_3\right)\vec{g}_2 + \left(-\tfrac{2}{3}x_1 + x_2\right)\vec{g}_2$$

$$\therefore \quad [\vec{x}]_\gamma = \begin{bmatrix} \tfrac{8}{3}x_1 \\ -\tfrac{5}{3}x_1 + x_3 \\ -\tfrac{2}{3}x_1 + x_2 \end{bmatrix}$$

Behold: $\underset{\gamma\leftarrow\beta}{P}[\vec{x}]_\beta = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} \tfrac{8}{3}x_1 \\ -\tfrac{5}{3}x_1 + x_3 \\ -\tfrac{2}{3}x_1 + x_2 \end{bmatrix}$$

Theorem 9.1 and Definition 9.1

Suppose that the set $\beta = \left\{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \right\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots c_n \vec{b}_n$. The constants $c_1 - c_n$ are called the $\beta -$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Theorem 9.2

Suppose that β and γ are both bases for $\mathbb{R}^n$ and that $T : \mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left([\vec{x}]_\beta\right) = [\vec{x}]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma \leftarrow \beta}{P}$ with the property that $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$.

Theorem 9.3

Suppose that β and γ are two ordered bases for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to β are known. Then the components of $\vec{x}$ relative to γ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from β to γ.

When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \mid \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if β is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to γ is simply γ^{-1}.