Definitions 8.1-8.3: Subspaces of $\mathbb{R}^n$

A ***subspace*** of the vector space $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ that contains the zero vector from $\mathbb{R}^n$ and is closed over vector addition and scalar multiplication.

A set of vectors, H, is ***closed over vector addition*** if and only if $\vec{u}+\vec{v}\in H\ \forall\ \vec{u},\vec{v}\in H$.

A set of vectors, H, is ***closed over scalar multiplication*** if and only if $c\vec{u}\in H\ \forall\ \vec{u}\in H, c\in\mathbb{R}$.

Theorem 8.1

The span of a set of vectors from $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Example 8.1

Illustrate Theorem 8.1 for the span of two vectors, $\vec{u}_1$ and $\vec{u}_2$.

Suppose that $\vec{m}, \vec{n} \in \text{Span}(\{\vec{u}_1, \vec{u}_2\})$.

Then $\exists$ scalars, x_1, x_2, y_1, y_2 such that

$\vec{m} = x_1\vec{u}_1 + x_2\vec{u}_2$ and $\vec{n} = y_1\vec{u}_1 + y_2\vec{u}_2$.

Closure over addition

$$\vec{m}+\vec{n} = (x_1\vec{u}_1 + x_2\vec{u}_2) + (y_1\vec{u}_1 + y_2\vec{u}_2)$$
$$= (x_1+y_1)\vec{u}_1 + (x_2+y_2)\vec{u}_2$$

$\vec{m}+\vec{n}$ is a linear combination $\rightarrow$ QED
of $\vec{u}_1$ + $\vec{u}_2$, it's in Span$(\{\vec{u}_1, \vec{u}_2\})$

Closure over scalar multiplication

$$k\vec{m} = k(x_1\vec{u}_1 + x_2\vec{u}_2)$$
$$= (kx_1)\vec{u}_1 + (kx_2)\vec{u}_2$$

QED

Definition 8.4

A set of linearly independent vectors that span a subspace of $\mathbb{R}^n$ is called a ***basis*** for that subspace of $\mathbb{R}^n$.

Example 8.2

Determine a basis for the set of vectors $\beta = \left\{ c_1 \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

Example 8.3

Determine a basis for the set of vectors $\gamma = \left\{ c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

8.2// β is obviously spanned by $\left\{ \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$ is obviously linearly independent.

$\therefore \left\{ \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$ is a basis for β.

8.3 γ is obviously spanned by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \right\}$.

But $(-2) \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so the set is <u>not</u> linearly independent – either vector alone spans all of γ.

$\therefore \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$ is <u>a</u> basis for γ

and $\left\{ \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \right\}$ is <u>a</u> basis for γ.

Definition 8.5 and Theorems 8.2-8.3

The ***column space*** of a matrix A is the set of all linear combinations of the column vectors of A. If A is an $m\times n$ matrix, then $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^m$. The pivot columns of A form a basis for the column space of A.

Example 8.4

Determine whether or not $[5,5,10]^{\mathrm{T}} \in \operatorname{col}(A)$ where $A=\begin{bmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{bmatrix}$.

$\begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}$ is in col(A) iff

$$x_1\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 6 \\ 14 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}$$

has at least one solution.

$$\left[\begin{array}{ccc|c} 3 & 4 & -2 & 5 \\ -1 & 6 & 4 & 5 \\ 5 & 14 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -14/11 & 0 \\ 0 & 1 & 5/11 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$0 \neq 1$, $\begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix} \notin \operatorname{col}(A)$.

$A\vec{x} = \vec{0}$

Example 8.5

Show that solutions to the equation $\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form a subspace of $\mathbb{R}^2$ (a) and then find a basis for that subspace. (b)

Suppose that $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

Then $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
$= \vec{0} + \vec{0}$
$= \vec{0}$

So the set is closed over vector addition.

Also, $A(k\vec{v}) = k(A\vec{v})$
$= k \cdot \vec{0}$
$= \vec{0}$

So the set is closed over scalar multiplication.

QED

b) $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}$

$\left[\begin{array}{cc|c} 3 & -7 & 0 \\ -6 & 14 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -7/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ Gen sol: $\begin{cases} x_1 = \frac{7}{3}x_2 \\ x_2 \text{ is free} \end{cases}$

Vectors in the null space of A can be written as $x_2\begin{bmatrix} 7 \\ 3 \end{bmatrix}$. $\therefore$ $\left\{\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right\}$ forms a basis for $\text{nul}(A)$.

Definition 8.6 and Theorems 8.4-8.5

The ***null space*** of a matrix A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. If A is an $m \times n$ matrix, then $\text{nul}(A)$ is a subspace of $\mathbb{R}^n$. A spanning set of the solution set to the homogenous system $A\vec{x} = \vec{0}$ forms a basis for the null space of A.

If $T(\vec{x}) = A\vec{x}$, the null space of A is the kernel of T.

Example 8.6

Let $A=\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}$. Determine whether $\begin{bmatrix}2 & 1 & 0\end{bmatrix}^{T}$ is an element of the column space or the null space of A.

$\begin{bmatrix}2\\1\\0\end{bmatrix}$ is obviously not in col(A), co(A) is from $\mathbb{R}^2$.

$$\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}\begin{bmatrix}2\\1\\0\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}2\\1\\0\end{bmatrix} \notin \text{nul}(A)$$

Example 8.7: Let $A=\begin{bmatrix}2 & 1 & 3 & 3\\ 1 & -1 & 3 & 1\\ -2 & 1 & -5 & 3\\ 3 & 2 & 4 & 2\end{bmatrix}$. Then $\text{RREF}(A)=\begin{array}{c}\begin{array}{cccc}x_1 & x_2 & x_3 & x_4\end{array}\\ \begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}$. $\begin{matrix}0\\0\\0\\0\end{matrix}$

a. Find a basis for the null space of A and explicitly show that it is indeed a basis for the null space of A.
b. Find a basis for the column space of A and explicitly show that it is indeed a basis for the column space of A.
c. State the rank of A.
d. State a basis for the row space of A.
e. State each row of A as a linear combination of the basis vectors stated in part (d).

a. The general solution to $A\vec{x}=\vec{0}$ is $\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$

So every vector in nul(A) can be written

as $x_3\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}$ $\left(\text{Check: } \begin{bmatrix}2 & 1 & 3 & 3\\ 1 & -1 & 3 & 1\\ -2 & 1 & -5 & 3\\ 3 & 2 & 4 & 2\end{bmatrix}\begin{bmatrix}-2\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right)$

$\left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}\right\}$ Spans nul(A) and it is obviously linearly indendent.

$\therefore \left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}\right\}$ is a basis for nul(A)

Example 8.7: Let $A = \begin{array}{cccc} P & P & & P \\ \left[\begin{matrix} 2 \\ 1 \\ -2 \\ 3 \end{matrix}\right. & \begin{matrix} 1 \\ -1 \\ 1 \\ 2 \end{matrix} & \begin{matrix} 3 \\ 3 \\ -5 \\ 4 \end{matrix} & \left.\begin{matrix} 3 \\ 1 \\ 3 \\ 2 \end{matrix}\right] \end{array}$. Then $\mathrm{RREF}(A) = \begin{array}{cccc} P & P & & P \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ -1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix}\right] \end{array}$.

b. Find a basis for the column space of A and explicitly show that it is indeed a basis for the column space of A.
c. State the rank of A.
d. State a basis for the row space of A.
e. State each row of A as a linear combination of the basis vectors stated in part (d).

b. A basis for the column space is:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$$

I need to demonstrate that $\vec{C}_1$, $\vec{C}_2$, & $\vec{C}_4$ are linearly independent and I need to demonstrate that $\vec{C}_3$ contributes nothing additional to the span (i.e. $\vec{C}_3$ is a linear combination of $\vec{C}_1, \vec{C}_2, \& \vec{C}_4$)

$$\left[\begin{array}{ccc|c|c} 2 & 1 & 3 & 0 & 3 \\ 1 & -1 & 1 & 0 & 3 \\ -2 & 1 & 3 & 0 & -5 \\ 3 & 2 & 2 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The only solution to $x_1\vec{C}_1 + x_2\vec{C}_2 + x_3\vec{C}_4 = \vec{0}$ is the trivial solution

and $2\vec{C}_1 + (-1)\vec{C}_2 + 0\vec{C}_4 = \vec{C}_3$ QED

Definition 8.7 and Theorems 8.6-8.7

The ***row space*** of a matrix A is the set of all linear combinations of the row vectors of A. If A is an $m \times n$ matrix, then $\mathrm{row}(A)$ is a subspace of $\mathbb{R}^n$. The pivot columns of A form a basis for the column space of A. The non-zero rows of $\mathrm{RREF}(A)$ for a basis for the row space of A.

Definition 8.8 and Theorem 8.9

The ***rank*** of a matrix A is the number of nonzero rows in $\mathrm{RREF}(A)$. The dimension of the column space of A is equal to the rank of A.

c. The rank of A is the number of pivot columns of A. $\operatorname{rank}(A) = 3$

d. A basis for the row space is:

$$\{[1,0,2,0], [0,1,-1,0], [0,0,0,1]\}$$

To demonstrate this, I need to show that the vectors are linearly independent and that each row of A is a linear combination of these three vectors.

$$x_1[1,0,2,0] + x_2[0,1,-1,0] + x_3[0,0,0,1] = [0,0,0,0]$$

$x_1 = 0$ $x_2 = 0$ $x_3 = 0$

$$2[1,0,2,0] + 1[0,1,-1,0] + 3[0,0,0,1] = [2,1,3,3]$$

check third column

$(2)(2) + (1)(-1) + (3)(0) = 3$ ✓

$$1[1,0,2,0] + (-1)[0,1,-1,0] + 1[0,0,0,1] = [1,-1,3,1]$$

CTC

$(1)(2) + (-1)(-1) + (1)(0) = 3$ ✓

$$-2[1,0,2,0] + 1[0,1,-1,0] + 3[0,0,0,1] = [-2,1,-5,3]$$

CTC

$(-2)(2) + (1)(-1) + (3)(0) = -5$ ✓

$$3[1,0,2,0] + 2[0,1,-1,0] + 2[0,0,0,1] = [3,2,4,2]$$

CTC

$(3)(2) + (2)(-1) + (2)(0) = 4$ ✓

Example 8.8

Let $A=\begin{bmatrix} 2 & 1 & -4 & 3 & -2 & -5 \\ -1 & 1 & 2 & 3 & 1 & -11 \\ 1 & 4 & -2 & -3 & -1 & 11 \\ -2 & -2 & 4 & -1 & 2 & -1 \end{bmatrix}$. Then $A\sim \begin{array}{cccccc} p & p & \text{free} & p & \text{free} & \text{free} \end{array}$ $\begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $A\vec{x}=\vec{0}$

a. What are the dimensions of the null space, column space, and row space of A and how do you know. What does the null-space dimension and column space dimension sum to?

b. State bases for the null space, column space, and row space of A.

a. Three non-zero rows in rref(A), ∴

rank(A) = 3, dim(row(A)) = 3, dim(col(A)) = 3

Since the solution to $A\vec{x}=\vec{0}$ has three free variables,

dim(nul(A)) = 3.

$$\dim(\mathrm{col}(A)) + \dim(\mathrm{nul}(A)) = 3+3$$
$$= 6 \text{ (\# of columns)}$$

b. A basis for the column space is:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ -1 \end{bmatrix} \right\}$$

A basis for the row space is:

$$\{ [1, 0, -2, 0, -1, 2], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, -3] \}$$

The general solution to $A\vec{x}=\vec{0}$ is

$$\begin{cases} x_1 = 2x_3 + x_5 + 2x_6 \\ x_2 = 0 \\ x_3 \text{ is free} \\ x_4 = 3x_6 \\ x_5 \text{ is free} \\ x_6 \text{ is free} \end{cases}$$

The null space vectors can be expressed as

$$x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

∴ A basis for the null space of A is

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example 8.9

Determine the dimension of $\operatorname{span}\left(\underbrace{\left\{\begin{bmatrix}1\\-1\\0\\4\\-2\end{bmatrix},\begin{bmatrix}-4\\4\\0\\-16\\8\end{bmatrix},\begin{bmatrix}-1\\2\\1\\3\\-2\end{bmatrix},\begin{bmatrix}1\\0\\1\\11\\-6\end{bmatrix},\begin{bmatrix}-2\\3\\1\\-1\\0\end{bmatrix}\right\}}_{\beta}\right)$.

Dimension

"How many vectors in a basis?"

The span of β is the Column Space of

$$A = \begin{bmatrix} 1 & -4 & -1 & 1 & -2 \\ -1 & 4 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 4 & -16 & 3 & 11 & -1 \\ -2 & 8 & -2 & -6 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & -4 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RANK} = 2$$

$\therefore \dim(\operatorname{col}(A)) = 2$

$\therefore \dim(\operatorname{span}(\beta)) = 2$

Example 8.10

Let $A = \begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$. The $A \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ Fill-in each of the following blanks.

The column space of M is a ___2___-dimensional subspace of $\mathbb{R}^{3}$.

The row space of M is a ___2___-dimensional subspace of $\mathbb{R}^{5}$.

The null space of M is a ___3___-dimensional subspace of $\mathbb{R}^{5}$.

The column space of M^{T} is a ___2___-dimensional subspace of $\mathbb{R}^{5}$.

The row space of M^{T} is a ___2___-dimensional subspace of $\mathbb{R}^{3}$.

The null space of M^{T} is a ___1___-dimensional subspace of $\mathbb{R}^{3}$.

M^T has three columns.

Theorems 8.8-8.10 and Definition 8.8

If one basis of a subspace of $\mathbb{R}^n$ contains m vectors, than every basis of **that** subspace contains m vectors. Additionally, any set of m linearly independent vectors from the subspace forms a basis for the subspace and any set of m vectors that spans the subspace forms a basis for the subspace. The ***dimension*** of such a subspace is m, i.e. the dimension of the subspace is the number of vectors in each basis for the subspace.

Theorem 8.10

If A is an $m \times n$ matrix, then $\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = n$.

Theorem 8.11: "Theorem 8" revisited

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x}=\vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x})=A\vec{x}$ is one-to-one.

g. The equation $A\vec{x}=\vec{b}$ has exactly one solution $\forall\, \vec{b}\in\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x})=A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A)\neq 0$

n. The columns of A form a basis $\mathbb{R}^n$.

o. $\mathrm{Col}(A)=\mathbb{R}^n$

p. $\dim(\mathrm{Col}(A))=n$

q. $\mathrm{Nul}(A)=\{\vec{0}\}$

r. $\dim(\mathrm{Nul}(A))=0$