Example 8.1

Illustrate Theorem 8.1 for the span of two vectors, $\vec{u}_1$ and $\vec{u}_2$.

Example 8.2

Determine a basis for the set of vectors $\left\{ c_1\begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

Example 8.3

Determine a basis for the set of vectors $\left\{ c_1\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$.

Example 8.4

Determine whether or not $[5,5,10]^{\mathrm{T}} \in \mathrm{col}(A)$ where $A = \begin{bmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{bmatrix}$.

Example 8.5

Show that solutions to the equation $\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form a subspace of $\mathbb{R}^2$ and then find a basis for that subspace.

Example 8.6

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & 6 \end{bmatrix}$. Determine whether $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$ is an element of the column space or the null space of A.

Example 8.7: Let $A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Find a basis for the null space of A and explicitly show that it is indeed a basis for the null space of A.
b. Find a basis for the column space of A and explicitly show that it is indeed a basis for the column space of A.
c. State the rank of A.
d. State a basis for the row space of A.
e. State each row of A as a linear combination of the basis vectors stated in part (d).

Example 8.8

Let $A=\begin{bmatrix} 2 & 1 & -4 & 3 & -2 & -5 \\ -1 & 1 & 2 & 3 & 1 & -11 \\ 1 & 4 & -2 & -3 & -1 & 11 \\ -2 & -2 & 4 & -1 & 2 & -1 \end{bmatrix}$. Then $A\sim\begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. What are the dimensions of the null space, column space, and row space of A and how do you know. What does the null-space dimension and column space dimension sum to?

b. State bases for the null space, column space, and row space of A.

Example 8.9

Determine the dimension of $\operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ 0 \\ -16 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 11 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right\}\right)$.

Example 8.10

Let $A=\begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$. The $A\sim\begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Fill-in each of the following blanks.

The column space of M is a ___________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The row space of M is a ______________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The null space of M is a ______________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The column space of M^{T} is a ___________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The row space of M^{T} is a ____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The null space of M^{T} is a ____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

Definitions 8.1-8.3: Subspaces of $\mathbb{R}^n$

A ***subspace*** of the vector space $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ that contains the zero vector from $\mathbb{R}^n$ and is closed over vector addition and scalar multiplication.

A set of vectors, H, is ***closed over vector addition*** if and only if $\vec{u}+\vec{v}\in H\ \forall\ \vec{u},\vec{v}\in H$.

A set of vectors, H, is ***closed over scalar multiplication*** if and only if $c\vec{u}\in H\ \forall\ \vec{u}\in H, c\in\mathbb{R}$.

Theorem 8.1

The span of a set of vectors from $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Definition 8.4

A set of linearly independent vectors that span a subspace of $\mathbb{R}^n$ is called a ***basis*** for that subspace of $\mathbb{R}^n$.

Definition 8.5 and Theorems 8.2-8.3

The ***column space*** of a matrix A is the set of all linear combinations of the column vectors of A. If A is an $m\times n$ matrix, then $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^m$. The pivot columns of A form a basis for the column space of A.

Definition 8.6 and Theorems 8.4-8.5

The ***null space*** of a matrix A is the set of all solutions to the equation $A\vec{x}=\vec{0}$. If A is an $m\times n$ matrix, then $\operatorname{nul}(A)$ is a subspace of $\mathbb{R}^n$. A spanning set of the solution set to the homogenous system $A\vec{x}=\vec{0}$ forms a basis for the null space of A.

Definition 8.7 and Theorems 8.6-8.7

The ***row space*** of a matrix A is the set of all linear combinations of the row vectors of A. If A is an $m\times n$ matrix, then $\operatorname{row}(A)$ is a subspace of $\mathbb{R}^n$. The pivot columns of A form a basis for the column space of A. The non-zero rows of $\operatorname{RREF}(A)$ for a basis for the row space of A.

Theorems 8.8-8.10 and Definition 8.8

If one basis of a subspace of $\mathbb{R}^n$ contains m vectors, than every basis of **<u>that</u>** subspace contains m vectors. Additionally, any set of m linearly independent vectors from the subspace forms a basis for the subspace and any set of m vectors that spans the subspace forms a basis for the subspace. The ***<u>dimension</u>*** of such a subspace is m, i.e. the dimension of the subspace is the number of vectors in each basis for the subspace.

Theorem 8.11: "Theorem 8" revisited

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***<u>nonsingular</u>***).

b. A is row equivalent to I_n .

c. A has n pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***<u>the trivial solution</u>***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall\, \vec{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A) \neq 0$

n. The columns of A form a basis $\mathbb{R}^n$.

o. $\mathrm{Col}(A) = \mathbb{R}^n$

p. $\dim(\mathrm{Col}(A)) = n$

q. $\mathrm{Nul}(A) = \{\vec{0}\}$

r. $\dim(\mathrm{Nul}(A)) = 0$