Example 7.1

Categorize each of the following function as one-to-one (injective), onto $\mathbb{R}$ (surjective), or both one-to-one and onto $\mathbb{R}$ (bijective): $y = \tan(x)$, $y = \sin(x)$, $y = \tan^{-1}(x)$, $y = x^3$.

Example 7.2

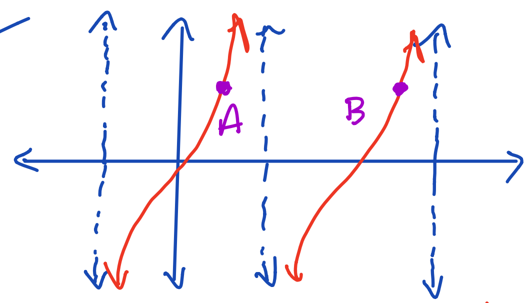
Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^2$

Definitions and a Theorem 7.1

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

7.1



$y = \tan(x)$

A & B have the same output, but different inputs, so tan is not one-to-one.

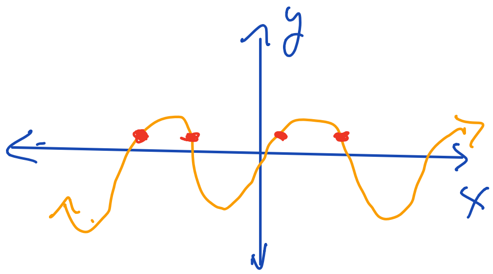
input	output
$\pi/4$	1
$\frac{5\pi}{4}$	1

not one-to-one

Every value in $\mathbb{R}$ is the output from at least one value in the domain; i.e. the range is all of $\mathbb{R}$.

$\therefore$ tan is onto $\mathbb{R}$

Tan is Surjective.

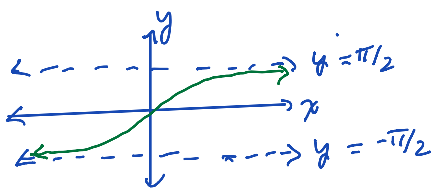


$y = \sin(x)$

Sin is not one-to-one

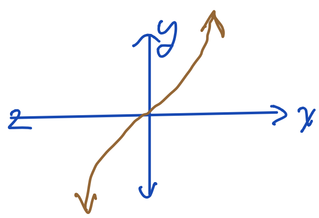
The range of sin is $[-1, 1]$,
So sin is not onto $\mathbb{R}$.

Sin is "no"-jective.



no repeat y-coordinates,
So $\tan^{-1}$ is one-to-one.
Range: $(-\pi/2, \pi/2)$
So $\tan^{-1}$ is not onto $\mathbb{R}$.

$\tan^{-1}$ is injective.



$y = x^3$ is both one-to-one and onto $\mathbb{R}$, so it is bijective.

7.2// $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^2$ iff $A\vec{x} = \vec{b}$ has a solution $\forall\ \vec{b} \in \mathbb{R}^2$.

$$\left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 2 & 4 & 16 & b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & c_1 \\ 0 & 1 & 5 & c_2 \end{array}\right]$$

Regardless of the values of b_1 & b_2, the relevant system always has the general solution.

$$\begin{cases} x_1 = 2x_3 + c_1 \\ x_2 = -5x_3 + c_2 \\ x_3 \text{ is free} \end{cases}$$

for some $c_1, c_2 \in \mathbb{R}$.

$\therefore T$ is onto $\mathbb{R}^2$.

Example 7.3

Prove that if T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

T cannot possibly be one-to-one because of the free variable in the solutions to $T(\vec{x}) = \vec{b}$. Also, $T: \mathbb{R}^3 \to \mathbb{R}^2$. The domain is larger than the codomain – T cannot possibly be one-to-one.

Example 7.4

Prove that the linear transformation T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

Hint: Prove the ***contrapositive statement***.

7.3 //

If T is a linear transformation, then for any $\vec{v} \in \text{dom}(T)$,

$$T(\vec{v} + \vec{0}) = T(\vec{v}) + T(\vec{0})$$

But, obviously $T(\vec{v} + \vec{0})$ also equals $T(\vec{v})$

Ergo $T(\vec{v}) + T(\vec{0}) = T(\vec{v})$,

Thus $T(\vec{v}) + T(\vec{0}) - T(\vec{v}) = T(\vec{v}) - T(\vec{v})$

$$\therefore \quad T(\vec{0}) = \vec{0}$$

QED

7.4 Contrapositive:

T is not one-to-one iff $T(\vec{x}) = \vec{0}$ has non-zero-vector solutions.

T is not one-to-one $\iff \exists\, \vec{u}, \vec{v} \in \text{dom}(T)$, $\vec{u} \neq \vec{v}$, but $T(\vec{u}) = T(\vec{v})$.

$\iff \exists\, \vec{u}, \vec{v} \in \text{dom}(T)$, $\vec{u} \neq \vec{v}$, such that $T(\vec{u}) - T(\vec{v}) = \vec{0}$.

$\iff \exists\, \vec{u}, \vec{v} \in \text{dom}(T)$, $\vec{u} \neq \vec{v}$, such that $T(\vec{u} - \vec{v}) = \vec{0}$.

not $\vec{0}$

QED

Example 7.5: Let $A = \begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Suppose that $T(\vec{x}) = A\vec{x}$. Find all values of λ that make T one-to-one. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

Theorem 7.2: A whole lot of equivalent properties (textbook "Theorem 8")

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall \vec{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^T is nonsingular.

m. $\det(A) \neq 0$

$T(\vec{x}) = A\vec{x}$ is one-to-one iff $\det(A) \neq 0$

(properties f and m of "Theorem 8".)

$$\det(A) = \begin{vmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \lambda + 2 \quad \text{(TI - Jake)}$$

$\therefore$ $T(\vec{x})$ is one-to-one iff $\lambda + 2 \neq 0$,

i.e. $\lambda \neq -2$.

Example 7.6

Demonstrate, both implicitly and explicitly, that the linear transformation T, given below, is neither one-to-one nor onto $\mathbb{R}^3$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + x_3 \\ x_1 + x_2 \\ 3x_1 + x_2 - x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2x_1 + x_3 \\ x_1 + x_2 \\ 3x_1 + x_2 - x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= A\vec{x}$$

where $A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$

$\det(A) = 0$ (Steve's head)

$\therefore$ We have implicity established that T is "no"-jective. (properties f, i, m of "Theorem 8")

To explicitly show that T is not injective (one-to-one), we need to show that two vectors $\vec{u} \neq \vec{v}$, have the property that $T(\vec{u}) = T(\vec{v})$. Let's solve $T(\vec{x}) = \vec{0}$.

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general Solution to $T(\vec{x}) = \vec{0}$ is

$$\begin{cases} x_1 = \frac{1}{2} x_3 \\ x_2 = -\frac{1}{2} x_3 \\ x_3 \text{ is free} \end{cases}$$

$$\therefore \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{QED}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

not one-to-one.

To show that T is not onto $\mathbb{R}^3$, we need to find a vector in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{v}$ has no Solutions.

I pick $\vec{v} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 7 \\ 1 & 1 & 0 & 11 \\ 3 & 1 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{Phew!}$$

There's a contradiction $(0 = 1)$.

QED