Example 7.1
Categorize each of the following function as one-to-one (injective), onto $\mathbb{R}$ (surjective), or both one-to-one and onto $\mathbb{R}$ (bijective): $y = \tan(x)$, $y = \sin(x)$, $y = \tan^{-1}(x)$, $y = x^3$.

Example 7.2

Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^2$ and also whether or not it is one-to-one

Example 7.3

Prove that if T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

Example 7.4

Prove that the linear transformation T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.
Hint: Prove the ***contrapositive statement***.

Example 7.5: Let $\mathrm{A} = \begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Suppose that $T(\vec{x}) = A\vec{x}$. Find all values of λ that make T one-to-one. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

Example 7.6
Demonstrate, both implicitly and explicitly, that the linear transformation T, given below, is neither one-to-one nor onto $\mathbb{R}^3$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + x_3 \\ x_1 + x_2 \\ 3x_1 + x_2 - x_3 \end{bmatrix}$$

Definitions and a Theorem 7.1

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

Theorem 7.2: A whole lot of equivalent properties (textbook "Theorem 8")

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall\ \vec{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A) \neq 0$