Example 6.1

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$? Describe the set of vectors whose images are $\vec{0}$.

> **Definitions 6.1-6.5: Transformations**
> A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T(\vec{x}) = \vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.
>
> $\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

read " T maps $\mathbb{R}^3$ into $\mathbb{R}^2$,

$$T\left(\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ is the image of $\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$ under T.

$\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$ is a preimage of $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ under T.

T maps $\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$ to $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

The domain of T is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$,

The range of T is the set of vectors in $\mathbb{R}^2$ whose second components are 0.

The set of preimages of $\vec{0}$ is the set $\left\{\begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} \middle|\, x_1, x_3 \in \mathbb{R}\right\}$.

Example 6.2

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

Definition 6.6: Linear Transformations

A ***linear transformation***, $T: \mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$ $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

$$T_1(\vec{u} + \vec{v}) = T_1\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} u_2 + v_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} v_2 \\ 0 \end{bmatrix}$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} cu_2 \\ 0 \end{bmatrix}$$

$$= c\begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$

$$= c\,T(\vec{u})$$

QED

$$T_2\left(3\begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}\right) = T_2\left(\begin{bmatrix} 15 \\ 12 \\ 21 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 12 \\ 1 \end{bmatrix}$$

$$\neq 3\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= 3\,T_2\left(\begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}\right)$$

QED

Example 6.3

Draw $T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)$ given that T is a linear transformation and the image under T for $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are those shown in Figure 1.

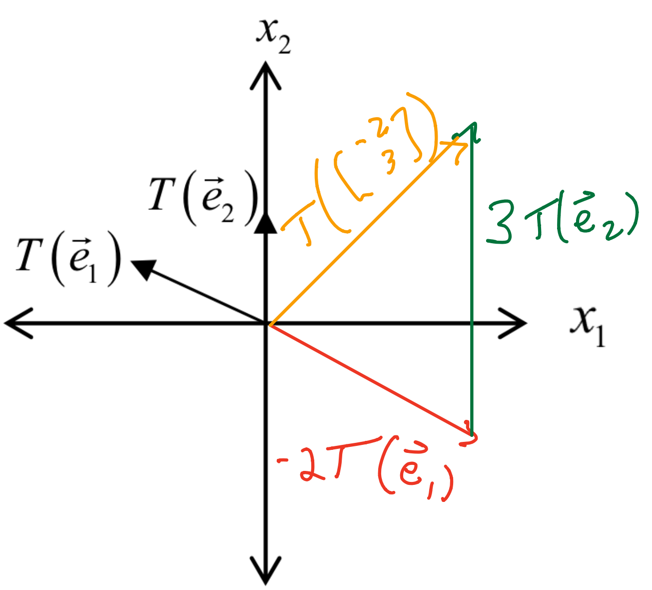


Figure 1: Transformation Vectors

$$T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = T\Big($$
$$= T\left(-2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= -2\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 3\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Example 6.4

Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^2$ and that $T(\vec{e}_1) = \begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix}5\\0\end{bmatrix}$.

a. Determine $T\left(\begin{bmatrix}-6 & 2 & 1\end{bmatrix}^{\mathrm{T}}\right)$.

b. Find a matrix, M, with the property that $T(\vec{x}) = M\,\vec{x} \quad \forall \vec{x} \in \mathbb{R}^3$.

$$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right) = -6\,T\Big(\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{e}_1}\Big) + 2\,T\Big(\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{e}_2}\Big) + 1\,T\Big(\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{e}_3}\Big)$$
$$= -6\begin{bmatrix}3\\-2\end{bmatrix} + 2\begin{bmatrix}-1\\4\end{bmatrix} + 1\begin{bmatrix}5\\0\end{bmatrix} \quad (\star)$$
$$= \begin{bmatrix}-15\\20\end{bmatrix}$$

From $(\star)$ we have:

$$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right) = \begin{bmatrix}3 & -1 & 5\\-2 & 4 & 0\end{bmatrix}\begin{bmatrix}-6\\2\\1\end{bmatrix}$$ i.e. $T(\vec{x}) = M\vec{x}$ where
$M = [T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)]$

Example 6.5

Suppose that $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example 6.6

Suppose that $T_1:\mathbb{R}^2 \to \mathbb{R}^2$ by reflecting vectors from $\mathbb{R}^2$ across the line $y = \sqrt{3}\,x$ and that $T_2:\mathbb{R}^2 \to \mathbb{R}^2$ by rotating vectors from $\mathbb{R}^2$ 30° in the counter-clockwise direction. Suppose further that $T:\mathbb{R}^2 \to \mathbb{R}^2$ by the rule $T(\vec{x}) = T_2(T_1(\vec{x}))$. Determine the linear transformation matrix for T (i.e. determine the matrix A such that $T(\vec{x}) = A\vec{x}\ \forall\ \vec{x} \in \mathbb{R}^2$). Finally, confirm the matrix A by tracking the vector $\vec{x} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ under T and then computing $A\vec{x}$.

6.5

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

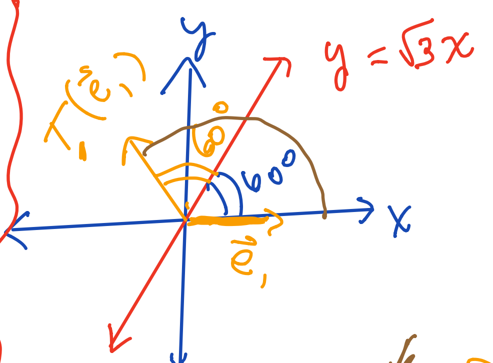
$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

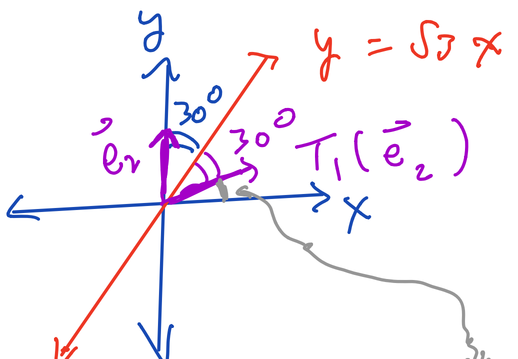
$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

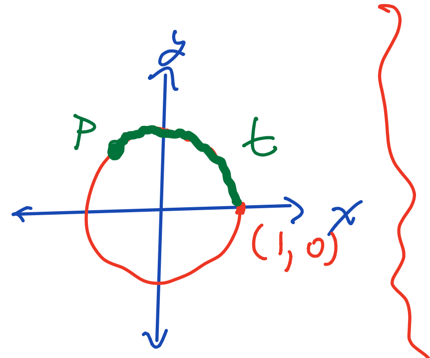
$T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$T(\vec{e}_1)$ $\quad$ $T(\vec{e}_2)$







P: $(\cos(t), \sin(t))$

$$T_1(\vec{e}_1) = \begin{bmatrix} \cos(120^{\circ}) \\ \sin(120^{\circ}) \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$T_1(\vec{e}_2) = \begin{bmatrix} \cos(30^{\circ}) \\ \sin(30^{\circ}) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

Scratch work for Example 6.6

6.6 //

at 150°

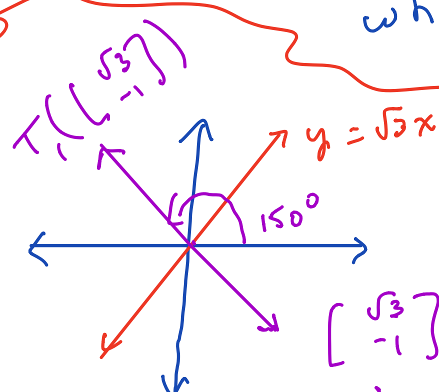
$$\begin{bmatrix}1\\0\end{bmatrix} \xrightarrow{T_1} \begin{bmatrix}-1/2\\ \sqrt{3}/2\end{bmatrix} \xrightarrow{T_2} \begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix} \quad T(\vec{e}_1)$$

$$\begin{bmatrix}0\\1\end{bmatrix} \xrightarrow{T_1} \begin{bmatrix}\sqrt{3}/2\\ 1/2\end{bmatrix} \xrightarrow{T_2} \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} \quad T(\vec{e}_2)$$

at 60°

$\therefore\quad T(\vec{x}) = A\vec{x}$

where $A = \begin{bmatrix}-\sqrt{3}/2 & 1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$



$T_1\left(\begin{bmatrix}\sqrt{3}\\-1\end{bmatrix}\right) = 2\begin{bmatrix}\cos(150°)\\ \sin(150°)\end{bmatrix}$

$= \begin{bmatrix}-\sqrt{3}\\1\end{bmatrix}$

$\begin{bmatrix}\sqrt{3}\\-1\end{bmatrix} = \vec{x}$

$|\vec{x}| = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= 2$

Scratch work

$$\begin{bmatrix}\sqrt{3}\\-1\end{bmatrix} \xrightarrow{T_1} \begin{bmatrix}-\sqrt{3}\\1\end{bmatrix} \xrightarrow{T_2} \begin{bmatrix}-2\\0\end{bmatrix}$$

$\uparrow\ T\left(\begin{bmatrix}\sqrt{3}\\-1\end{bmatrix}\right)$

Punch line

$T(\vec{x}) = A\vec{x}$

$= \begin{bmatrix}-\sqrt{3}/2 & 1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}\sqrt{3}\\-1\end{bmatrix}$

$= \begin{bmatrix}-2\\0\end{bmatrix}$ ✓

Example 6.7

Determine a general matrix for the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors from $\mathbb{R}^2$ through an angle θ.

Example 6.8

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the $x_1 x_2$-plane by 60^{o} in the clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

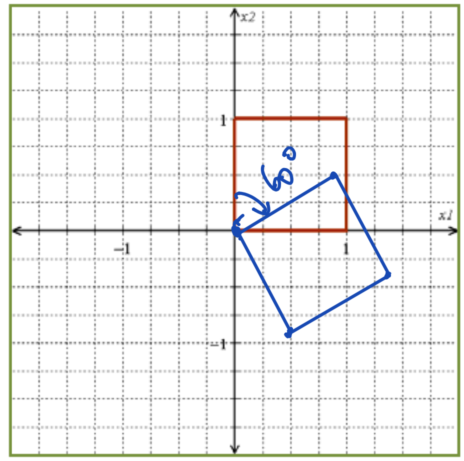
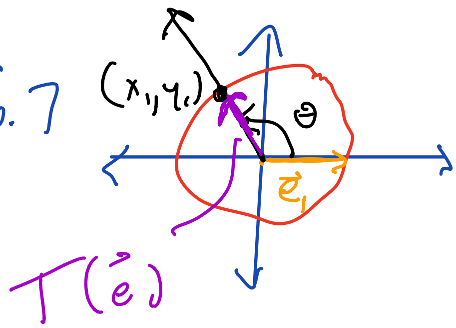
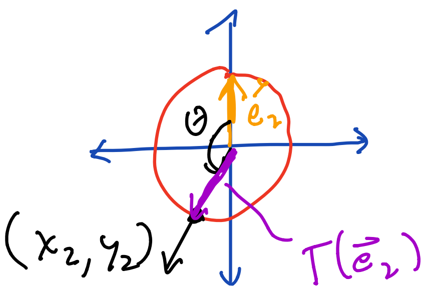


Figure 2: Rotated "unit square"

6.7



$$x_1 = \cos(\theta)$$
$$y_1 = \sin(\theta)$$



$$\begin{aligned} x_2 &= \cos(\theta + 90^\circ) \\ &= \cos(\theta)\cos(90^\circ) \\ &\quad - \sin(\theta)\sin(90^\circ) \\ &= -\sin(\theta) \end{aligned} \qquad \begin{aligned} y_2 &= \sin(\theta + 90^\circ) \\ &= \sin(\theta)\cos(90^\circ) \\ &\quad + \cos(\theta)\sin(90^\circ) \\ &= \cos(\theta) \end{aligned}$$

$$\therefore \; T(\vec{x}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

6.8) $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$

$$= \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} \approx \begin{bmatrix} .5 \\ -.9 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \approx \begin{bmatrix} .9 \\ .5 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.4 \\ -.4 \end{bmatrix}$$

Example 6.9

Determine the linear transformation matrix for T given that $T\left(\begin{bmatrix}4\\4\end{bmatrix}\right)=\begin{bmatrix}16\\-4\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=\begin{bmatrix}2\\5\end{bmatrix}$.

We need to express $\vec{e}_1$ & $\vec{e}_2$ in terms of $\begin{bmatrix}4\\4\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$, i.e. we need to solve

$$x_1\begin{bmatrix}4\\4\end{bmatrix}+x_2\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$ and

$$y_1\begin{bmatrix}4\\4\end{bmatrix}+y_2\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\left[\begin{array}{cc|cc}4&1&1&0\\4&3&0&1\end{array}\right]\sim\left[\begin{array}{cc|cc}1&0&3/8&-1/8\\0&1&-1/2&1/2\end{array}\right]$$

$\therefore$

$$\begin{aligned}T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)&=T\left(\tfrac{3}{8}\begin{bmatrix}4\\4\end{bmatrix}+(-\tfrac{1}{2})\begin{bmatrix}1\\3\end{bmatrix}\right)\\&=\tfrac{3}{8}T\left(\begin{bmatrix}4\\4\end{bmatrix}\right)+(-\tfrac{1}{2})T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)\\&=\tfrac{3}{8}\begin{bmatrix}16\\-4\end{bmatrix}+(-\tfrac{1}{2})\begin{bmatrix}2\\5\end{bmatrix}\\&=\begin{bmatrix}5\\-4\end{bmatrix}\end{aligned}$$

$$\begin{aligned}T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)&=T\left(-\tfrac{1}{8}\begin{bmatrix}4\\4\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}1\\3\end{bmatrix}\right)\\&=-\tfrac{1}{8}\begin{bmatrix}16\\-4\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}2\\5\end{bmatrix}\\&=\begin{bmatrix}-1\\3\end{bmatrix}\end{aligned}$$

$\therefore\ T(\vec{x})=A\vec{x}$ where $A=\begin{bmatrix}5&-1\\-4&3\end{bmatrix}$

Check

$T\left(\begin{bmatrix}4\\4\end{bmatrix}\right)=\begin{bmatrix}5&-1\\-4&3\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}16\\-4\end{bmatrix}$ ✓ $\quad T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=\begin{bmatrix}5&-1\\-4&3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$ ✓