Example 6.1

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$? Describe the set of vectors whose images are $\vec{0}$.

Example 6.2

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

Example 6.3

Draw $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$ given that T is a linear transformation and the image under T for $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are those shown in Figure 1.

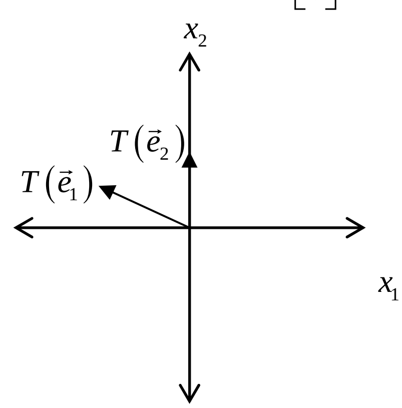

Figure 1: Transformation Vectors

Example 6.4

Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^2$ and that $T(\vec{e}_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

a. Determine $T\left(\begin{bmatrix} -6 & 2 & 1 \end{bmatrix}^{\mathrm{T}}\right)$.

b. Find a matrix, M, with the property that $T(\vec{x}) = M\,\vec{x} \;\; \forall \;\; \vec{x} \in \mathbb{R}^3$.

Example 6.5

Suppose that $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example 6.6

Suppose that $T_1:\mathbb{R}^2 \to \mathbb{R}^2$ by reflecting vectors from $\mathbb{R}^2$ across the line $y = \sqrt{3}\,x$ and that $T_2:\mathbb{R}^2 \to \mathbb{R}^2$ by rotating vectors from $\mathbb{R}^2$ 30^{o} in the counter-clockwise direction. Suppose further that $T:\mathbb{R}^2 \to \mathbb{R}^2$ by the rule $T(\vec{x}) = T_2(T_1(\vec{x}))$. Determine the linear transformation matrix for T (i.e. determine the matrix A such that $T(\vec{x}) = A\vec{x}\ \forall\ \vec{x} \in \mathbb{R}^2$). Finally, confirm the matrix A by tracking the vector $\vec{x} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ under T and then computing $A\vec{x}$.

Example 6.7

Determine a general matrix for the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors from $\mathbb{R}^2$ through an angle θ.

Example 6.8

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the $x_1 x_2$-plane by 60^{o} in the clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

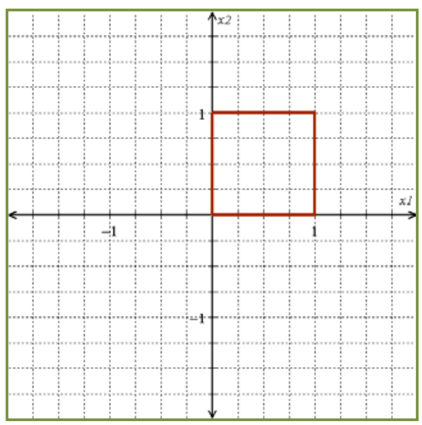

Figure 2: Rotated "unit square"

Example 6.9

Determine the linear transformation matrix for T given that $T\left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ -4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Definitions 6.1-6.5: Transformations
A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T\left(\vec{x}\right)=\vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Definition 6.6: Linear Transformations
A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T\left(\vec{u}+\vec{v}\right)=T\left(\vec{u}\right)+T\left(\vec{v}\right) \text{ and } T\left(c\vec{u}\right)=cT\left(\vec{u}\right) \quad \forall\, \vec{u},\vec{v}\in\mathbb{R}^n \text{ and } c\in\mathbb{R}$$

Theorem 6.1
If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n$ under the linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ completely determines **all** of the images under T.

Theorem 6.2
Every transformation of form $T\left(\vec{x}\right)=A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T\left(\vec{x}\right)=A\vec{x}$.