Ex 5.1

(Sum across the first row)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{j=1}^{2} \left[a_{1,j}\, C_{1,j} \right]$$

$$= a_{11} C_{11} + a_{12} C_{12}$$

$$= a_{11} \cdot (-1)^{1+1} |a_{22}| + a_{12} \cdot (-1)^{1+2} |a_{21}|$$

$$= a_{11} \cdot 1 \cdot a_{22} + a_{12} \cdot (-1) \cdot a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

Let's repeat going down the second column

$$\begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{i=1}^{2} \left[a_{i,2}\, C_{i,2} \right]$$

$$= a_{12} C_{12} + a_{22} C_{22}$$

$$= a_{12} \cdot (-1)^{1+2} \cdot |a_{21}| + a_{22} \cdot (-1)^{2+2} \cdot |a_{11}|$$

$$= a_{12} \cdot (-1) \cdot a_{21} + a_{22} \cdot 1 \cdot a_{11}$$

$$= a_{11} a_{22} - a_{12} a_{21} \checkmark$$

Ex 5.2 $A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$

(second row)

$$\det(A) = \sum_{j=1}^{3} \left[a_{2,j} C_{2,j} \right]$$

$$= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$= 3 \cdot (-1)^{2+1} \begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + 1 \cdot (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + (-4) \cdot (-1)^{2+3} \begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix}$$

$$= (3) \cdot (-1) \cdot (18 - (-9)) + (1)(1)(6 - (-1)) + (-4)(-1)(18 - 6)$$

$$= -81 + 7 + 48$$

$$= -26$$

(Sum down first column)

$$\det(A) = \sum_{i=1}^{3} \left[a_{i,1} C_{i,1} \right]$$

$$= a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31}$$

$$= (2)(-1)^{1+1} \begin{vmatrix} 1 & -4 \\ 9 & 3 \end{vmatrix} + (3)(-1)^{2+1} \begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + (1)(-1)^{3+1} \begin{vmatrix} 6 & -1 \\ 1 & -4 \end{vmatrix}$$

$$= (2)(1)(3-(-36)) + (3)(-1)(18-(-9)) + (1)(1)(-24-(-1))$$

$$= 78 - 81 - 23$$

$$= -26 \checkmark$$

Ex. 5.3//

$$B = \begin{bmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{bmatrix}$$

$$\det(B) = \sum_{i=1}^{4} [b_{i,3} C_{i,3}]$$

$$= b_{13}C_{13} + b_{23}C_{23} + b_{33}C_{33} + b_{43}C_{43}$$

$$= 0C_{13} + (-2)\cdot(-1)^{2+3} \cdot \begin{vmatrix} 3 & 9 & -1 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix} + 0C_{33} + 0C_{43}$$

$$= (-2)(-1)\cdot\Big((3)(-1)^{1+1}\begin{vmatrix} 5 & 4 \\ -6 & 6 \end{vmatrix} + (2)(-1)^{2+1}\begin{vmatrix} 9 & -1 \\ -6 & 6 \end{vmatrix}$$
$$+ (0)(-1)^{3+1}\begin{vmatrix} 9 & -1 \\ 5 & 4 \end{vmatrix}\Big)$$

$$= 2\big(3(30+24) - 2(54-6) + 0\big)$$

$$= 132$$

Ex 5.4// $\vec{u} = [1, 7, -3]$ & $\vec{v} = [3, 0, 5]$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 7 & -3 \\ 3 & 0 & 5 \end{vmatrix}$$

$$= \hat{i}\cdot(-1)^{1+1}\begin{vmatrix} 7 & -3 \\ 0 & 5 \end{vmatrix} + \hat{j}\cdot(-1)^{1+2}\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + \hat{k}\cdot(-1)^{1+3}\begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix}$$

$$= \hat{i}(35-0) - \hat{j}(5+9) + \hat{k}(0-21)$$

$$= 35\hat{i} - 14\hat{j} - 21\hat{k}$$

$$= [35, -14, -21]$$

Ex 5.5 // $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ note: $\det(B) = 5$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ note: $\det(I) = 1$

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$I \xrightarrow{R_1 \leftrightarrow R_2} A$ $\Big\{$ $\det(A) = 0 - 1 = -1 = -\det(I)$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$B \xrightarrow{R_1 \leftrightarrow R_2} AB$ $\Big\{$ $\det(AB) = -2 - 3 = -5 = -\det(B)$

b. $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$I \xrightarrow{3R_1 \to R_1} A$ $\Big\{$ $\det(A) = 3 - 0 = 3 = 3\det(I)$

$$AB = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 2 & 1 \end{bmatrix}$$

$B \xrightarrow{3R_1 \to R_1} AB$ $\Big\{$ $\det(AB) = 9 + 6 = 15 = 3(5) = 3\det(B)$

c. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$I \xrightarrow{-2R_2 \to R_2} A$

$$\begin{aligned} \det(A) &= -2 - 0 \\ &= -2 \\ &= -2 \det(I) \end{aligned}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ -4 & -2 \end{bmatrix} \end{aligned}$$

$B \xrightarrow{-2R_2 \to R_2} AB$

$$\begin{aligned} \det(AB) &= -6 - 4 \\ &= -10 \\ &= -2(5) \\ &= -2 \det(B) \end{aligned}$$

d. $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$I \xrightarrow{-3R_1 + R_2 \to R_2} A$

$$\begin{aligned} \det(A) &= 1 - 0 \\ &= 1 \\ &= \det(I) \end{aligned}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ -7 & 4 \end{bmatrix} \end{aligned}$$

$B \xrightarrow{-3R_1 + R_2 \to R_2} AB$

$$\begin{aligned} \det(AB) &= 12 - 7 \\ &= 5 \\ &= \det(B) \end{aligned}$$

e. $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$I \xrightarrow{4R_2 + R_1 \to R_1} A$

$$\begin{aligned} \det(A) &= 1 - 0 \\ &= 1 \\ &= \det(I) \end{aligned}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 3 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

$B \xrightarrow{4R_2 + R_1 \to R_1} AB$

$$\begin{aligned} \det(AB) &= 11 - 6 \\ &= 5 \\ &= \det(B) \end{aligned}$$

Ex 5.6

$$A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$$

effect

$$\begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 9 & 3 \\ 3 & 1 & -4 \\ 2 & 6 & -1 \end{bmatrix} B \qquad \det(B) = -\det(A)$$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & -12 & -7 \end{bmatrix} C \qquad \det(C) = \det(B) = -\det(A)$$

$$-\frac{1}{26}R_2 \to R_2 \quad \begin{bmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & -12 & -7 \end{bmatrix} D \qquad \det(D) = \frac{-1}{26}\det(C) = \frac{1}{26}\det(A)$$

$$12R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} E \qquad \det(E) = \det(D) = \frac{1}{26}\det(A)$$

$$\therefore \quad \det(A) = 26\det(E)$$

(sum down Column 1)

Shown for development purposes

$$= \left(1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 1/2 \\ 0 & -1 \end{vmatrix} + 0 + 0\right) 26$$

$$= \left(1 \cdot \left((1)(-1) - (0)(1/2)\right)\right) \cdot 26$$

$$= (1)(1)(-1) \cdot 26$$

$$= -26$$

Effect

$$B = \begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 9 & 2 \\ 3 & 1 & -1 & 4 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{bmatrix} C \quad \Big\} \ \det(C) = -\det(B)$$

$-3R_1 + R_2 \to R_2$
$-8R_1 + R_3 \to R_3$
$-4R_1 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & -3 & -71 & -9 \\ 0 & 2 & -30 & -9 \end{bmatrix} D \quad \Big\} \ \det(D) = \det(C) = -\det(B)$$

$3R_2 + R_3 \to R_3$
$-2R_2 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & -155 & -15 \\ 0 & 0 & 26 & -5 \end{bmatrix} E \quad \Big\} \ \det(E) = \det(D) = -\det(B)$$

$-\frac{1}{155} R_3 \to R_3$

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & 28 & -2 \\ 0 & 0 & 1 & 3/31 \\ 0 & 0 & 26 & -5 \end{bmatrix} F \quad \Big\} \ \det(F) = \frac{-1}{155}\det(E) = \frac{1}{155}\det(B)$$

$-26R_3 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & 28 & -2 \\ 0 & 0 & 1 & 3/31 \\ 0 & 0 & 0 & -233/31 \end{bmatrix} G \quad \Big\} \ \det(G) = \det(F) = \frac{1}{155}\det(B)$$

$$\therefore \det(B) = 155 \det(G)$$
$$= 155\left[(1)(1)(1)(-233/31)\right)$$
$$= -1165$$

Ex 5.7) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$\det(AB) = \begin{vmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{vmatrix}$$

$$= (ax+bz)(cy+dw) - (ay+bw)(cx+dz)$$

$$= \cancel{acxy} + adwx + bcyz + \cancel{bdwz} - \cancel{acxy} - adyz - bcwx - \cancel{bdwz}$$

$$= adwx - adyz + bcyz - bcwx$$

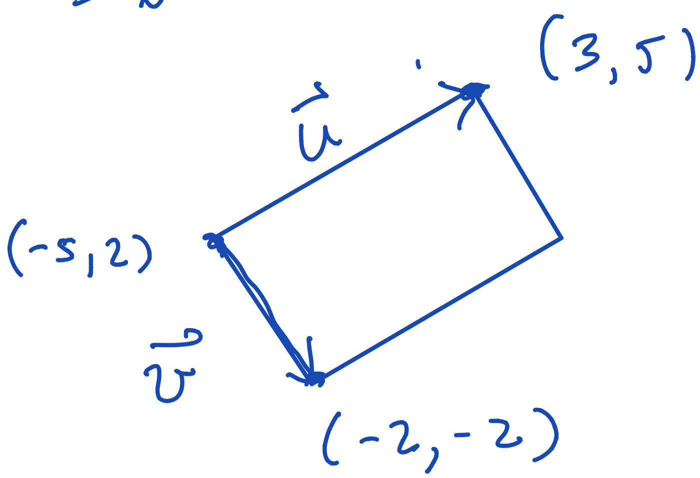
$$= ad(wx - yz) + bc(yz - wx)$$

$$= ad\underbrace{(wx - yz)} - bc\underbrace{(wx - yz)}$$

$$= (ad - bc)\underbrace{(wx - yz)}$$

$$= \det(A)\det(B)$$

Ex 5.8



$$\vec{u} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\text{Area} = \left| \begin{vmatrix} 8 & 3 \\ 3 & -4 \end{vmatrix} \right| = |-41| = 41$$

determinant

absolute value

Ex 5.9

$$\begin{bmatrix} 3 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}$$

Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} 1 & 1 \\ -11 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ -3 & 4 \end{vmatrix}} = \frac{15}{15} = 1$$

$$x_2 = \frac{\begin{vmatrix} 3 & 1 \\ -3 & -11 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ -3 & 4 \end{vmatrix}} = \frac{-30}{15} = -2$$

The solution is $(1, -2)$

b. $$\begin{bmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -28 \\ -23 \\ 5 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} -28 & -1 & 4 \\ -23 & 0 & 3 \\ 5 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{vmatrix}} = \frac{7}{-1} = -7$$

$$x_2 = \frac{\begin{vmatrix} 2 & -28 & 4 \\ 2 & -33 & 3 \\ 1 & 5 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{vmatrix}} = \frac{-2}{-1} = 2$$

$\therefore$ The Solution is $(-7, 2, -3)$.

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & -28 \\ 2 & 0 & -33 \\ 1 & 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{vmatrix}} = \frac{3}{-1} = -3$$

Ex 5.10

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = -18 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = 10 \qquad C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3 \qquad C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6 \qquad C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$
$$= (2)(10) + (2)(-2) + 3(-6)$$
$$= -2$$

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$$
$$= \frac{1}{-2}\begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -1.5 & -5 \\ -5 & 1 & 3 \\ -2 & .5 & 1 \end{bmatrix}$$