Example 5.1

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

Example 5.2

Use cofactors along the second row to find $\det(A)$ where $A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

Example 5.3

Evaluate $\det(B)$ where $B = \begin{bmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{bmatrix}$.

Example 5.4

Use a determinant to find $\vec{u} \times \vec{v}$ where $\vec{u} = [1, 7, -3]$ and $\vec{v} = [3, 0, 5]$.

Example 5.5: Let $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$.

For each of the following matrices (each identified as A), describe the row operation that was affected upon I_2 to create A. Then find $\det(A)$ and compare its value to $\det(I_2)$. Next, find AB and describe the difference between it and B. Finally, compare the values of $\det(AB)$ and $\det(B)$.

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c.** $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ d. $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ e. $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Example 5.6

Determine $\det(A)$ and $\det(B)$ after first manipulating the matrices into upper triangular form.

$$A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{bmatrix}$$

Example 5.7

Illustrate Theorem 5.3 using $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Example 5.8

Find the area of the parallelogram outline in Figure 1.

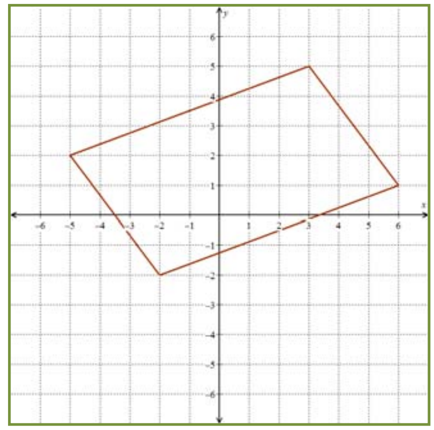

Figure 1: A parallelogram

Example 5.9

Use Cramer's Rule to find the solutions to each of the following matrix equations.

a. $\begin{bmatrix} 3 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -28 \\ -23 \\ 5 \end{bmatrix}$.

Example 5.10

Use the determinant and adjoint to find A^{-1} where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

Definition 5.1: Determinants (of square matrices)

$$\det\left(\left[a_{11}\right]\right) = \left| a_{11} \right| = a_{11}$$

For a square matrix, A, with two or more rows we define the cofactor of entry $\underline{\boldsymbol{a_{ij}}}$, $\boldsymbol{C_{ij}}$, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the i^{th} row and j^{th} column from A. Then using any row of A or any column of A:

$$\det(\mathrm{A}) = \sum_{j=1}^{n}\left[a_{ij}\, C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}\, C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed i^{th} row of A whereas in the second formula we are summing along a fixed j^{th} column of A.

Definition 5.2: Elementary Matrices

An ***elementary matrix*** is a matrix that can be created from an identity matrix via one elementary row operation.

Theorem 5.1: Elementary Row Operations and Determinants

Suppose that A and B are square matrices of equal dimension; suppose further that B can be created from A via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of A to a different row of A, then $\det(B) = \det(A)$
- if the operation is swapping two rows of A, then $\det(B) = -\det(A)$
- if the operation is multiplying a row of A by the real number k, then $\det(B) = k \cdot \det(A)$.

Definitions 5.3-5.4 and Theorem 5.2

An ***upper triangular matrix*** is a matrix where every entry *below* the main diagonal is zero.
A ***lower triangular matrix*** is a matrix where every entry *above* the main diagonal is zero.

The determinant of any $n \times n$ triangular matrix B is given by the formula $\det(B) = \prod_{i=1}^{n} b_{ii}$.

Theorems 5.3 and 5.4

- If A and B are like-sized square matrices, then $\det(AB) = \det(A)\det(B)$.
- If A is any square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

Theorem 5.5: A Little Geometry

$\left\| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\|$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Theorem 5.6: Cramer's Rule

Consider the matrix equation $A\vec{x} = \vec{b}$ where A is a nonsingular square matrix. Define $A_i(\vec{b})$ to be the matrix derived from A by replacing the ith colum of A with $\vec{b}$. Then the solution to $A\vec{x} = \vec{b}$ can be found using the formula $x_i = \dfrac{\det\left(A_i(\vec{b})\right)}{\det(A)}$ for each x_i.

Theorem 5.7: An algorithm for finding inverse matrices

The matrix of cofactors of a square matrix *A* is the matrix that results from replacing each of its entries by their corresponding cofactors.

The Adjoint (Adjugate) of A is the transpose of *A*'s matrix of cofactors.

The inverse of a nonsingular square matrix *A* is $A^{-1} = \dfrac{1}{\det(A)}\mathrm{Adj}(A)$.

Please note that this implies that the matrix *A* is nonsingular if and only if $\det(A) \neq 0$.This also implies, albeit less directly, that the square system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.