Ex 4.1// a) B is 5×3 (# of rows $\times$ # of columns)

b) $b_{32} = 11$, $b_{23} = 0$, $b_{41} = -18$, b_{14} isn't

Ex 4.2// i) column vector iii) (row) vector

ii) not a vector iv) (row) vector

Ex 4.3//

$A+B$ cannot be computed / dimensional mismatch (A is 3×2 and B is 2×3)

$$B - 2C = \begin{bmatrix} 4 & 8 & -2 \\ -5 & -1 & 6 \end{bmatrix} - 2 \begin{bmatrix} -1 & -2 & 6 \\ 3 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & -2 \\ -5 & -1 & 6 \end{bmatrix} - \begin{bmatrix} -2 & -4 & 12 \\ 6 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 & -14 \\ -11 & 3 & 10 \end{bmatrix}$$

$$C + A^T = \begin{bmatrix} -1 & -2 & 6 \\ 3 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 6 \\ 2 & 0 & -9 \end{bmatrix}$$

Ex 4.4// i and iii

Ex 4.5 $\quad A$ is 3×2, B is 2×4

$$\underbrace{3\times \overbrace{2 \quad 2}^{\text{match}} \times 4}_{}$$

$3\times 2 \quad 2\times 4 \quad$ the product is 3×4

match

good to go

$$AB = \begin{bmatrix} A_{R_1}\cdot B_{C_1} & A_{R_1}\cdot B_{C_2} & A_{R_1}\cdot B_{C_3} & A_{R_1}\cdot B_{C_4} \\ A_{R_2}\cdot B_{C_1} & A_{R_2}\cdot B_{C_2} & A_{R_2}\cdot B_{C_3} & A_{R_2}\cdot B_{C_4} \\ A_{R_3}\cdot B_{C_1} & A_{R_3}\cdot B_{C_2} & A_{R_3}\cdot B_{C_3} & A_{R_3}\cdot B_{C_4} \end{bmatrix}$$

$$A_{R_1}\cdot B_{C_1} = \begin{bmatrix} 2 & 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$= (2)(-1) + (1)(-3)$$

$$= -5$$

$$A_{R_3}\cdot B_{C_4} = \begin{bmatrix} 3 & -2 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$= (3)(7) + (-2)(-5)$$

$$= 31$$

$$AB = \begin{bmatrix} -5 & 11 & -4 & 9 \\ -22 & 42 & -24 & -2 \\ 3 & -1 & 8 & 31 \end{bmatrix}$$

Ex 4.7 $\underset{2\times 5}{A}\ \underset{5\times 2}{B}$ is 2×2

$\checkmark$ match

$$AB = \begin{bmatrix} 2 & 1 & 0 & 8 & -5 \\ 1 & 3 & 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 10 & 1 \\ 2 & 4 \\ 0 & 3 \\ -6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 46 & -14 \\ 47 & 1 \end{bmatrix}$$

Ex 4.7 $\underset{2\times 5}{A}\ \underset{3\times 2}{B}$ is noncomputable,

$\checkmark$ no go

4.8 In Example 4.7, AB is not computable but BA is (and it's 3×5) so it's impossible that in general $AB = BA$.

Even is A & B were, say, 3×3, it's highly unlikely that $AB = BA$

4.9 //

$$\begin{bmatrix} 2 & 6 & -3 \\ -1 & 5 & 4 \\ 9 & 12 & -7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -3 \\ -1 & 5 & 4 \\ 9 & 12 & -7 \end{bmatrix}$$

4.10 //

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}\begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the 3×3 identity matrix. Where defined, $AI = A$ and $IA = A$.

$AB = I$, A and B are inverse matrices. $A = B^{-1}$, $B = A^{-1}$

4.11 $CD = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$$D = C^{-1}$$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$

$$AA^{-1} = I \Rightarrow \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 3y_1 & 2x_2 + 3y_2 \\ 4x_1 + 5y_1 & 4x_2 + 5y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives us two systems of equations

$$\begin{cases} 2x_1 + 3y_1 = 1 \\ 4x_1 + 5y_1 = 0 \end{cases} \qquad \begin{cases} 2x_2 + 3y_2 = 0 \\ 4x_2 + 5y_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 5 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 5 & 1 \end{array}\right]$$

Two - FER - ONE

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & x_1 & x_2 \\ 0 & 1 & y_1 & y_2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -5/2 & 3/2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

Check

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Ex 4.12

$$A = \begin{bmatrix} -12 & -5 & -9 \\ -4 & -2 & -4 \\ -8 & -4 & -6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -12 & -5 & -9 & 1 & 0 & 0 \\ -4 & -2 & -4 & 0 & 1 & 0 \\ -8 & -4 & -6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -R_1 \to R_1 \\ -R_2 \to R_2 \\ -R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 12 & 5 & 9 & -1 & 0 & 0 \\ 4 & 2 & 4 & 0 & -1 & 0 \\ 8 & 4 & 6 & 0 & 0 & -1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 4 & 2 & 4 & 0 & -1 & 0 \\ 12 & 5 & 9 & -1 & 0 & 0 \\ 8 & 4 & 6 & 0 & 0 & -1 \end{array}\right]$$

$$\begin{array}{l} -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 4 & 2 & 4 & 0 & -1 & 0 \\ 0 & -1 & -3 & -1 & 3 & 0 \\ 0 & 0 & -2 & 0 & 2 & -1 \end{array}\right]$$

$$\begin{array}{l} 2R_3 + R_1 \to R_1 \\ -\frac{3}{2}R_3 + R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 4 & 2 & 0 & 0 & 3 & -2 \\ 0 & -1 & 0 & -1 & 0 & 3/2 \\ 0 & 0 & -2 & 0 & 2 & -1 \end{array}\right]$$

$$2R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -2 & 3 & 1 \\ 0 & -1 & 0 & -1 & 0 & 3/2 \\ 0 & 0 & -2 & 0 & 2 & -1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{4}R_1 \to R_1 \\ -R_2 \to R_2 \\ -\frac{1}{2}R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3/4 & 1/4 \\ 0 & 1 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -1 & 1/2 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1/2 & 3/4 & 1/4 \\ 1 & 0 & -3/2 \\ 0 & -1 & 1/2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & -6 \\ 0 & -4 & 2 \end{bmatrix}$$

Check

$$-1 \begin{bmatrix} 12 & 5 & 9 \\ 4 & 2 & 4 \\ 8 & 4 & 6 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & -6 \\ 0 & -4 & 2 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \checkmark$$

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -1R_1+R_2 \to R_2 \\ -6R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right]$$

$$-4R_2+R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$R_3+R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$R_2+R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\therefore \quad B^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

Check//

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Ex 4.13//

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$-\frac{c}{a}R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & -\frac{bc}{a}+d & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow{aR_2 \to R_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$\xrightarrow{-\frac{b}{ad-bc}R_2 + R_1 \to R_1} \left[\begin{array}{cc|cc} a & 0 & 1+\frac{bc}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$\begin{array}{l} \frac{1}{a}R_1 \to R_1 \\ \frac{1}{ad-bc}R_2 \to R_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

million dollar formula (only applies to 2x2)

$$\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{14-12}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3/2 & 7/2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 5 & -2 \end{bmatrix}^{-1} = \frac{1}{2-15}\begin{bmatrix} -2 & -3 \\ -5 & -1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -4 & -1 \end{bmatrix}^{-1} = \frac{1}{-2+12}\begin{bmatrix} -1 & -3 \\ 4 & 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -1 & -3 \\ 4 & 2 \end{bmatrix}$$

Ex 4.14

$$\begin{cases} 4x_1 + 3x_2 = 20 \\ -2x_1 + 5x_2 = -36 \end{cases}$$

Background:

$$A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\Rightarrow I\vec{x} = A^{-1}\vec{b}$$
$$\Rightarrow \vec{x} = A^{-1}\vec{b}$$

All this assumes that A is invertable

Back to the problem...

The given system can be written as

$A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 20 \\ -36 \end{bmatrix}$

$$\therefore \vec{x} = A^{-1}\vec{b}$$
$$= \frac{1}{20+6}\begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 20 \\ -36 \end{bmatrix}$$
$$= \frac{1}{26}\begin{bmatrix} 208 \\ -104 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

How can A not be invertable

(i.e. how can A be singular)

Check this out

$$\begin{bmatrix} 2 & -7 \\ 6 & -21 \end{bmatrix}^{-1} = \frac{1}{\underbrace{-42+42}_{\text{woh woh}}}$$

Let $A=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix}$. Find, by hand, AB, BA, $(AB)^T$, $(BA)^T$, A^TB^T, B^TA^T, A^{-1}, B^{-1}, $(A^{-1})^{-1}$, $(AB)^{-1}$, $(BA)^{-1}$, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$, $(A^{-1})^T$, and $(A^T)^{-1}$. See what equals what.

$$AB = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix} = \begin{bmatrix}13 & 4\\ 23 & 7\end{bmatrix}$$

$$BA = \begin{bmatrix}-4 & -1\\ 7 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} = \begin{bmatrix}-11 & -17\\ 20 & 31\end{bmatrix}$$

$$(AB)^T = \begin{bmatrix}13 & 23\\ 4 & 7\end{bmatrix} \qquad (BA)^T = \begin{bmatrix}-11 & 20\\ -17 & 31\end{bmatrix}$$

$$B^TA^T = \begin{bmatrix}-4 & 7\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} = \begin{bmatrix}13 & 23\\ 4 & 7\end{bmatrix}$$

$$A^TB^T = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}-4 & 7\\ -1 & 2\end{bmatrix} = \begin{bmatrix}-11 & 20\\ -17 & 31\end{bmatrix}$$

$$A^{-1} = \frac{1}{10-9}\begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix} = \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}$$

$$B^{-1} = \frac{1}{-8+7}\begin{bmatrix}2 & 1\\ -7 & -4\end{bmatrix} = \begin{bmatrix}-2 & -1\\ 7 & 4\end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{10-9}\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{91-92}\begin{bmatrix}7 & -4\\ -23 & 15\end{bmatrix} = \begin{bmatrix}-7 & 4\\ 23 & -15\end{bmatrix}$$

$$(BA)^{-1} = \frac{1}{-341+340}\begin{bmatrix}31 & 17\\ -20 & -11\end{bmatrix} = \begin{bmatrix}-31 & -17\\ 20 & 11\end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix}-2 & -1\\ 7 & 4\end{bmatrix}\begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix} = \begin{bmatrix}-7 & 4\\ 23 & -15\end{bmatrix}$$

$$A^{-1}B^{-1} = \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2 & -1\\ 7 & 4\end{bmatrix} = \begin{bmatrix}-31 & -17\\ 20 & 11\end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix} \qquad (A^T)^{-1} \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}$$

Conclusion: $(AB)^T = B^TA^T$, $(BA)^T = A^TB^T$, $(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1}$, $(BA)^{-1} = A^{-1}B^{-1}$, $(A^{-1})^T = (A^T)^{-1}$

All of the above are always true for compatible invertible matrices.

We also saw that $A^T = A$. This cannot be true for all matrices because $B^T \neq B$. Matrices that equal their transpose are called Symmetric matrices.