Example 3.1

a. $$\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$$

$$\Rightarrow \begin{cases} 9x_1 + x_2 = 0 \\ 5x_1 - 2x_2 = 0 \end{cases}$$

$\therefore$ The system is homogeneous

b. $$4\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 4x_1 + 2x_1 = 0 \\ 4x_2 = -10 \\ 4x_3 + 2x_2 = 0 \end{cases}$$

not homogeneous

c. Aw, a kitty cat!

3.2

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 - 3x_2 + 7x_3 = 0 \\ -3x_1 + x_2 - 7x_3 = 0 \\ 4x_1 + 8x_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ -3 & 1 & -7 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The general solution is $\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$

$\therefore$ Solutions can be written in vector form as $\begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix}$

which in turn is equivalent to $x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ The solution set is Span$\left(\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$

This solution set is a line through the origin.

Ex 3.3//

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 9 \\ -3 & 1 & -7 & -10 \\ 4 & 0 & 8 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

general solution : $\begin{cases} x_1 = -2x_3 + 3 \\ x_2 = x_3 - 1 \\ x_3 \text{ is free} \end{cases}$

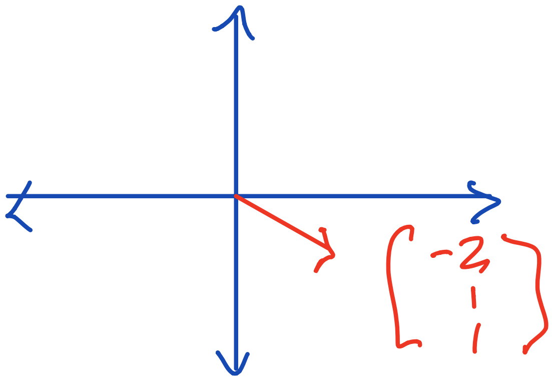
$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 + 3 \\ x_3 - 1 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

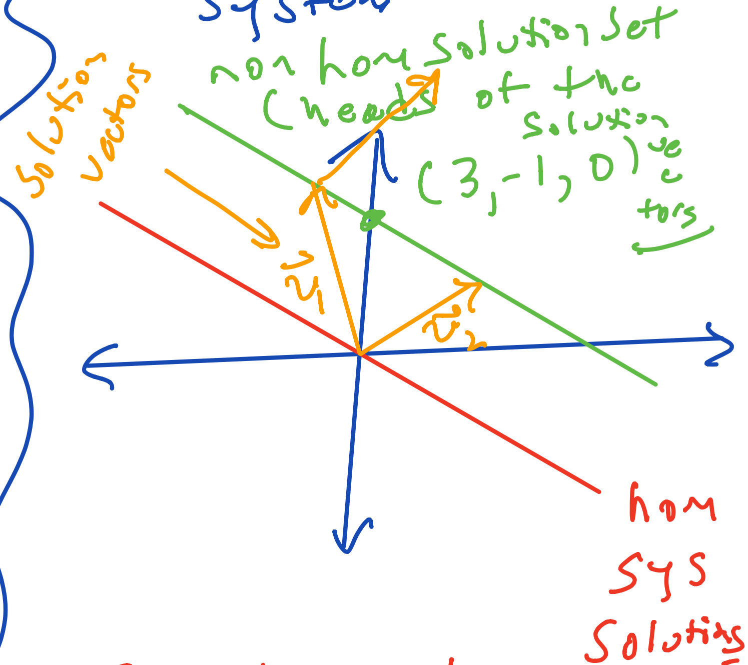
This is the solution set to the analogous (same coefficients) homogeneous system shifted so that it passes through the point $(3, -1, 0)$ rather than the origin.

Homogeneous system Solution set.



If we add two multiples of this solution vector we get another multiple of this solution vector. We say that this solution set is closed over scalar multiplication and vector addition.

non homogeneous system



Solutions to the hom system can be drawn onto the red line.

Solutions to the non-homogeneous system cannot be drawn onto any common line.

The head of $\vec{v}_1 + \vec{v}_2$ does not lie on the green line. Solution sets to non-homogeneous systems are not closed over either scalar multiplication or vector addition.

Non homogeneous systems do not have spanning sets for their solution set.

3.4//

$\{ 3x_1 - 12x_2 - 6x_3 = 0$ (homogeneous)

The general solution is

$$\begin{cases} x_1 = 4x_2 + 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

So solutions can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

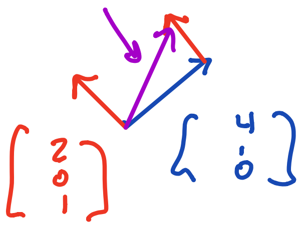
$$= x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Every vector in the solution set can written as a linear combination

of $\begin{bmatrix}4\\1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\\1\end{bmatrix}$ and any linear combination of those two vectors <u>is</u> in the solution set.

The solution set is Span$\left(\left\{\begin{bmatrix}4\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}\right\}\right)$

$\begin{bmatrix}6\\1\\1\end{bmatrix}$ is also in the solution set.



Collectively, this solution set forms a plane through the origin

$\{\, 3x_1 - 12x_2 - 6x_3 = -15$ (nonhomogeneous system)

Solutions can be written as

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_2\begin{bmatrix}4\\1\\0\end{bmatrix} + x_3\begin{bmatrix}2\\0\\1\end{bmatrix} + \begin{bmatrix}-5\\0\\0\end{bmatrix}$$

This is the homogeneous solution shifted from the origin to the point $(-5, 0, 0)$, but it only shifts the <u>heads</u> of the vectors. The vectors themselves do <u>not</u> lie on a common plan. Two solution vectors add to a not-a-solution vector.

3.5) $$A = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$$

The column vectors of A are linearly dependent if and only if:

$$x_1 \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has non-trivial solutions (not all x_1, x_2, x_3 equal to zero).

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -3 & 4 & -2 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

QED

QED (Quod Erat Deductium)
It means "it has been demonstrated"
It's written in lieu of a formal conclusion.

In this case it's appropriate because the RREF(A) indicates the presence of free variables in the solution set.

which means that the vector equation has "infinitely many solutions", all but one of which is non-trivial.

The general solution is $\begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$

So an explicit illustration of the linear dependency is

$$-2\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Ex 3.6

$$A = \begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$$

The column vectors of A are linearly independent iff the only solution to

$$x_1\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} a \\ a \\ -2 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 3 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$ (trivial solution)

$$\left[\begin{array}{ccc|c} 2 & a & -2 & 0 \\ 3 & a & 3 & 0 \\ -1 & -2 & a & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 3 & a & 3 & 0 \\ 2 & a & -2 & 0 \end{array}\right]$$

$$3R_1 + R_2 \to R_2, \quad 2R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & a-6 & 3a+3 & 0 \\ 0 & a-4 & 2a-2 & 0 \end{array}\right]$$

$$-\frac{a-4}{a-6}R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} -1 & -2 & a & 0 \\ 0 & a-6 & 3a+3 & 0 \\ 0 & 0 & * & 0 \end{array}\right]$$

$$* = -\frac{a-4}{a-6}(3a+3) + (2a-2)$$

$\therefore$ The implied vector equation has a unique solution (the trivial solution) iff

$$-\frac{a-4}{a-6}(3a+3) + (2a-2) \neq 0$$

$$(a-6) \cdot \left[-\frac{a-4}{a-6}(3a+3) + (2a-2)\right] \neq (a-6) \cdot 0$$

$$-(a-4)(3a+3) + (a-6)(2a-2) \neq 0$$

$$-a^2 - 5a + 24 \neq 0$$

$$a^2 + 5a - 24 \neq 0$$

$$(a+8)(a-3) \neq 0$$

$\therefore$ The column vectors of A are linearly independent iff $a \neq -8$ and $a \neq 3$.

Check

$$\left[\begin{array}{ccc|c} 2 & -8 & -2 & 0 \\ 3 & -8 & 3 & 0 \\ -1 & -2 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftarrow \text{non trivial Solutions.}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & 0 \\ 3 & 3 & 3 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftarrow \text{non trivial Solutions}$$

Just to be safe try $a=1$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 3 & 1 & 3 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ only trivial Solutions.}$$

Ex. 3.7

$\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent $\iff$ $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$ has non-trivial solutions.

$\iff$ $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$ for some pair of numbers, x_1 & x_2, not both 0.

$\iff$ $\vec{v}_1 = -\frac{x_2}{x_1}\vec{v}_2$

and/or $\vec{v}_2 = -\frac{x_1}{x_2}\vec{v}_1$

QED

Ex. 3.8 $\quad A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}$

Three vectors span all of $\mathbb{R}^3$ iff the vectors are linearly independent (Thm. 3.4)

$$\left[\begin{array}{ccc|c} -4 & 1 & 6 & 0 \\ -1 & 1 & 4 & 0 \\ 7 & -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ The column vectors of A are linearly independent, as the only solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ is trivial solution.

$\therefore$ The column vectors of A <u>do</u> span all of $\mathbb{R}^3$.

Ex 3.9

$\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. A set of two vectors can span at most a plane, hence β cannot span $\mathbb{R}^3$. Expand β so that it does span $\mathbb{R}^3$.

$$\vec{v}_3 = \begin{bmatrix} 5(2) + 7(1) \\ 5(1) + 7(-8) \\ \text{not } 5(-4) + 7(3) \end{bmatrix} = \begin{bmatrix} 17 \\ -51 \\ 0 \end{bmatrix}$$

independent

$$\left[\begin{array}{ccc|c} 2 & 1 & 17 & 0 \\ 1 & -8 & -51 & 0 \\ -4 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 17 & 0 \\ 1 & -8 & -51 & 0 \\ -4 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

dependent

3.10 $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ cannot possibly be linearly independent because three vectors from $\mathbb{R}^2$ are never linearly independent (Thm. 3.2)

$\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$ cannot possibly be linearly independent because it contains $\vec{0}$. To wit:

$$0 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + e^{\pi} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

non-trivial solution.