

Example 3.1

Determine whether or not each of the following is a homogeneous system of equations.

a.
$$\begin{cases} 2x_1 + 4x_2 = 3x_2 - 7x_1 \\ 5x_1 - 6 = 2x_2 - 6 \end{cases}$$

b.
$$4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2 \begin{bmatrix} x_1 \\ 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Example 3.2**

Describe the solution set to the homogeneous system
$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 3.3

Compare the solution set from the last example with the solution set to the nonhomogeneous system

$$\begin{bmatrix} 2 & -3 & 7 \\ -3 & 1 & -7 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ 12 \end{bmatrix}.$$

Example 3.4

Compare and contrast the solution sets to the homogeneous system $\{3x_1 - 12x_2 - 6x_3 = 0\}$ and the nonhomogeneous system $\{3x_1 - 12x_2 - 6x_3 = -15\}$.

Example 3.5

Show that the column vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & -2 \\ 3 & -5 & 1 \end{bmatrix}$ are linearly dependent.

Example 3.6

Determine the values of a that make the column vectors of $\begin{bmatrix} 2 & a & -2 \\ 3 & a & 3 \\ -1 & -2 & a \end{bmatrix}$ linearly independent.

Example 3.7

Prove Theorem 3.3.

Example 3.8

Determine whether or not the columns of A span \mathbb{R}^3 where $A = \begin{bmatrix} -4 & 1 & 6 \\ -1 & 1 & 4 \\ 7 & -1 & -3 \end{bmatrix}.$

Example 3.9

Consider the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix} \right\}$. Explain why the set cannot possibly span \mathbb{R}^3 . Afterwards, add a

vector to the set so that it does span \mathbb{R}^3 .

Example 3.10

Determine which of the following sets are linearly independent; explain!

a. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 21 \end{bmatrix} \right\}$

Definitions 3.1-3.3 and Theorem 3.1

A **homogeneous system of equations** is a system that can be written in the form $A\vec{x} = \vec{0}$.

Every homogeneous system of equations has at least one solution ($\vec{0}$); $\vec{0}$ is called the **trivial solution** to a homogeneous system of equations. Any other solution to a homogeneous system of equations is called a **nontrivial solution**.

Definitions 3.4 and 3.5

The set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is said to be **linearly independent** if and only if the only solution to the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = \vec{0}$ is $c_1 = c_2 = \dots = c_n = 0$. The set is said to be **linearly dependent** if there is a nontrivial solution to the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = \vec{0}$.

Theorems 3.2-3.5

- A set of vectors containing more than n vectors from \mathbb{R}^n must be linearly dependent.
- A set of **two** vectors is linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other vector.
- A set of n vectors from \mathbb{R}^n span \mathbb{R}^n if and only if the set is linearly independent.
- If A is an $m \times n$ matrix, then the columns of A span \mathbb{R}^m if and only if A has a pivot position in every row.