**Example 2.1**

Simplify $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$ and illustrate the process on Figure 1.

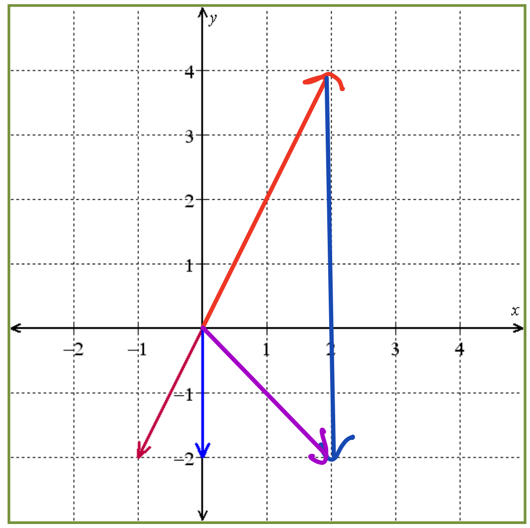



**Figure 1**: $-2\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}$

**Example 2.2**

Let $\vec{a}_1=\begin{bmatrix}2\\-4\end{bmatrix}$, $\vec{a}_2=\begin{bmatrix}-5\\-1\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}12\\-13\end{bmatrix}$. Express $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$ .

**Example 2.3**

Let $A=\begin{bmatrix}3&-1&5\\-2&0&-4\\-1&4&2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}3\\h\\2\end{bmatrix}$. Find the value of $h$ if $\vec{b}$ is in the span of the columns of $A$.

**Example 2.4**

Let $A=\begin{bmatrix}2&-6&-1\\0&5&2\end{bmatrix}$, $B=\begin{bmatrix}1&-3\\4&3\\-2&7\end{bmatrix}$, $\vec{u}=\begin{bmatrix}10\\2\end{bmatrix}$, and $\vec{v}=\begin{bmatrix}-2\\4\\8\end{bmatrix}$. Find each of the following products (where possible): $A\vec{u}$ , $A\vec{v}$ , $B\vec{u}$ , and $B\vec{v}$ .

**Example 2.5**

Write $\begin{bmatrix}-1&2&5\\8&-2&0\\1&-3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7\\11\\-3\end{bmatrix}$ as a vector equation.

**Example 2.6**

Write the system $\begin{cases}2x_1 - 3x_2 + x_3 - 2x_4 = 0\\ 5x_1 - x_2 + x_4 = 9\end{cases}$ as a matrix equation of form $A\vec{x}=\vec{b}$ .

Ex 2.1: $(-2)\begin{bmatrix}-1\\-2\end{bmatrix}+3\begin{bmatrix}0\\-2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}0\\-6\end{bmatrix}$

$$=\begin{bmatrix}2\\-2\end{bmatrix}$$

Ex. 2.2 $\vec{a}_1=\begin{bmatrix}2\\-4\end{bmatrix}$, $\vec{a}_2=\begin{bmatrix}-5\\-1\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}12\\-13\end{bmatrix}$

$\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$ if and only if there exists scalars such that $x_1\vec{a}_1+x_2\vec{a}_2=\vec{b}$.

$x_1\begin{bmatrix}2\\-4\end{bmatrix}+x_2\begin{bmatrix}-5\\-1\end{bmatrix}=\begin{bmatrix}12\\-13\end{bmatrix}$ (vector equation)

$\Rightarrow\begin{bmatrix}2x_1\\-4x_1\end{bmatrix}+\begin{bmatrix}-5x_2\\-x_2\end{bmatrix}=\begin{bmatrix}12\\-13\end{bmatrix}$

$\Rightarrow\begin{bmatrix}2x_1-5x_2\\-4x_1-x_2\end{bmatrix}=\begin{bmatrix}12\\-13\end{bmatrix}$

$\Rightarrow$ The system $\begin{cases}2x_1-5x_2=12\\-4x_1-x_2=-13\end{cases}$

$$\left[\begin{array}{cc|c}2&-5&12\\-4&-1&-13\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&7/2\\0&1&-1\end{array}\right]$$

$\therefore\ \vec{b}=\frac{7}{2}\vec{a}_1+(-1)\vec{a}_2$

$\frac{7}{2}$ is the weight on $\vec{a}_1$ and $-1$ is the weight on $\vec{a}_2$

Check $\frac{7}{2}\begin{bmatrix}2\\-4\end{bmatrix}+(-1)\begin{bmatrix}-5\\-1\end{bmatrix}=\begin{bmatrix}12\\-13\end{bmatrix}$ ✓

Ex 2.3 //

$$A = \begin{bmatrix} 3 & -1 & 5 \\ -2 & 0 & -4 \\ -1 & 4 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$$

$\vec{b}$ is in the span of the columns of $A$ if and only if $\vec{b}$ can be written as a linear combination of the column vectors from $A$; i.e. there is at least one ordered triple solution to

$$x_1 \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & h \\ -1 & 4 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ -2 & 0 & -4 & h \\ 3 & -1 & 5 & 3 \end{array}\right]$$

$$\begin{array}{l} -2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & h-4 \\ 0 & 11 & 11 & 9 \end{array}\right]$$

$$\tfrac{11}{8} R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & h-4 \\ 0 & 0 & 0 & \tfrac{11}{8}h + \tfrac{7}{2} \end{array}\right]$$

$\therefore$ $\vec{b}$ is in the span of the column vectors of $A$ if and only if

$$\frac{11}{8} h + \frac{7}{2} = 0 \; ; \text{ i.e. } h = -28/11$$

CK //

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & -28/11 \\ -1 & 4 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 14/11 \\ 0 & 1 & 1 & 9/11 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftarrow \text{no contradiction}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & 7 \\ -1 & 4 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Leftarrow \text{contradiction}$$

Ex 2.4

Dimensional Analysis

$A\vec{u}$ $\quad 2\times 3 \quad 2\times 1$ $\quad$ $A\vec{u}$ does not exist

non-match

$A\vec{v}$ $\quad 2\times 3 \quad 3\times 1$ $\quad$ $A\vec{v}$ is defined and its dimensions are $2\times 1$

Match

$B\vec{u}$ $\quad 3\times 2 \quad 2\times 1$ $\quad$ $B\vec{u}$ is $3\times 1$

match

$B\vec{v}$ $\quad 3\times 2 \quad 3\times 1$ $\quad$ no $B\vec{v}$

non-match

$$A\vec{v} = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$$

$$= (-2)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (4)\begin{bmatrix} -6 \\ 5 \end{bmatrix} + (8)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(2) + (4)(-6) + (8)(-1) \\ (-2)(0) + (4)(5) + (8)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -36 \\ 36 \end{bmatrix}$$

$$B\vec{u} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$= (10)\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 46 \\ -6 \end{bmatrix}$$

Ex 2.5

$$\begin{bmatrix} -1 & 2 & 5 \\ 8 & -2 & 0 \\ 1 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix} \Rightarrow x_1\begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$$

Matrix equation $\qquad$ vector equation.

Ex 2.6

$$\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 + x_4 = 9 \end{cases} \Rightarrow x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

vector equation

system of equations

$$\Rightarrow \begin{bmatrix} 2 & -3 & 1 & -2 \\ 5 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

matrix equation

Ex 2.7 :

$$C_2H_6 + O_2 \to CO_2 + H_2O$$

Let's let $x_1 - x_4$ represent the number of molecules in a balanced equation.

Specifically $x_1\, C_2H_6 + x_2\, O_2 \to x_3\, CO_2 + x_4\, H_2O$

Let's represent each molecule by the vector

$$\begin{bmatrix} \text{\# of carbon atoms} \\ \text{\# of oxygen atoms} \\ \text{\# of hydrogen atoms} \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

our system is:
$$\begin{cases} 2x_1 - x_3 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \\ 6x_1 - 2x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 6 & 0 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -7/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{array}\right]$$

general solution :
$$\begin{cases} x_1 = \frac{1}{3}x_4 \\ x_2 = \frac{7}{6}x_4 \\ x_3 = \frac{2}{3}x_4 \\ x_4 \text{ is free} \end{cases}$$

you can't have fractional molecules, so I choose to let $x_4 = 6$.

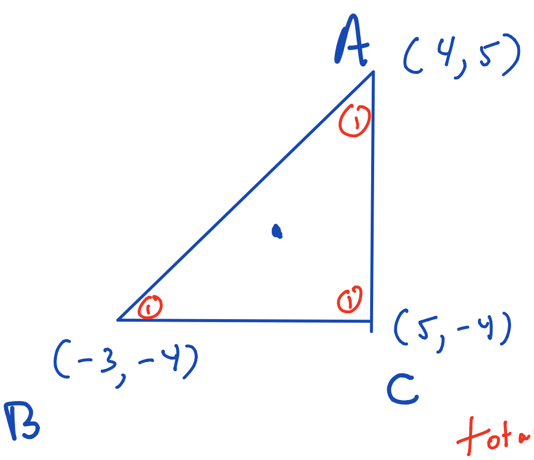
$\therefore$ A balanced equation is $2C_2H_6 + 7O_2 \rightarrow 4CO_2 + 6H_2O$

Check: H: $2(6) = 6(2)$ ✓

O: $7(2) = 4(2) + 6(1)$ ✓

C: $2(2) = 4(1)$ ✓

Ex 2.8




The center of mass is located at

$$\frac{1}{3}\left[1\begin{bmatrix}-3\\-4\end{bmatrix} + 1\begin{bmatrix}4\\5\end{bmatrix} + 1\begin{bmatrix}5\\-4\end{bmatrix}\right]$$

total mass

$\therefore$ The center of mass is at $(2,-1)$

Ex: Let $m_1$, $m_2$, and $m_3$ be, respectively, the total mass at the points A, B, and C.

Our goal is to move the center of mass to $(3,-2)$.

$$\frac{1}{12}\left(m_1\begin{bmatrix}4\\5\end{bmatrix}+m_2\begin{bmatrix}-3\\-4\end{bmatrix}+m_3\begin{bmatrix}5\\-4\end{bmatrix}\right)=\begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{cases}\frac{1}{3}m_1-\frac{1}{4}m_2+\frac{5}{12}m_3=3\\ \frac{5}{12}m_1-\frac{1}{3}m_2-\frac{1}{3}m_3=-2\\ m_1+m_2+m_3=12\end{cases}$$

$$\Longrightarrow\begin{cases}4m_1-3m_2+5m_3=36\\ 5m_1-4m_2-4m_3=-24\\ m_1+m_2+m_3=12\end{cases}$$

$$\left[\begin{array}{ccc|c}4&-3&5&36\\5&-4&-4&-24\\1&1&1&12\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&8/3\\0&1&0&8/3\\0&0&1&20/3\end{array}\right]$$

$\therefore$ To move the C. of M. to $(3,-2)$ we want to add $\frac{5}{3}g$ to points a and b and $\frac{17}{3}g$ to poin C.