

Example 2.1

Simplify $-2\begin{bmatrix} -1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and illustrate the process on

Figure 1.

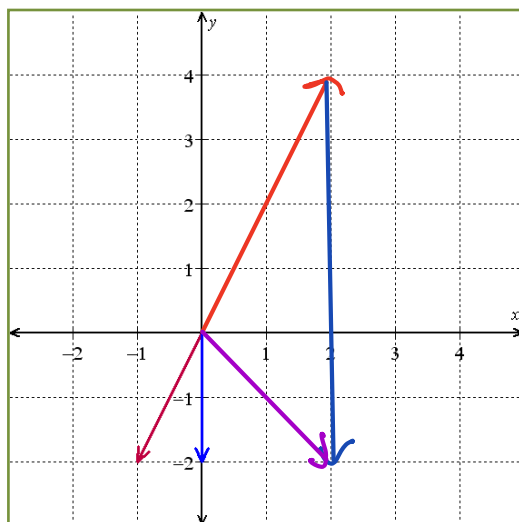


Figure 1: $-2\begin{bmatrix} -1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 2.2

Let $\vec{a}_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$. Express \vec{b} as a linear combination of \vec{a}_1 and \vec{a}_2 .

Example 2.3

Let $A = \begin{bmatrix} 3 & -1 & 5 \\ -2 & 0 & -4 \\ -1 & 4 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$. Find the value of h

if \vec{b} is in the span of the columns of A .

Example 2.4

Let $A = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$. Find each of the following products (where possible): $A\vec{u}$, $A\vec{v}$, $B\vec{u}$, and $B\vec{v}$.

Example 2.5

Write $\begin{bmatrix} -1 & 2 & 5 \\ 8 & -2 & 0 \\ 1 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$ as a vector equation.

Example 2.6

Write the system $\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 + x_4 = 9 \end{cases}$ as a matrix equation of form $A\vec{x} = \vec{b}$.

$$\text{Ex 2.1: } (-2) \begin{bmatrix} -1 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} \\ = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\text{Ex. 2.2} \quad \vec{a}_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

\vec{b} is a linear combination of \vec{a}_1 and \vec{a}_2 if and only if there exists scalars such that $x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$.

$$x_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix} \quad (\text{vector equation})$$

$$\Rightarrow \begin{bmatrix} 2x_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 & -5x_2 \\ -4x_1 & -x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix}$$

$$\Rightarrow \text{The system } \begin{cases} 2x_1 - 5x_2 = 12 \\ -4x_1 - x_2 = -13 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -5 & 12 \\ -4 & -1 & -13 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & -1 \end{array} \right]$$

$$\therefore \vec{b} = \frac{7}{2} \vec{a}_1 + (-1) \vec{a}_2$$

$\frac{7}{2}$ is the weight on \vec{a}_1 and -1 is the weight on \vec{a}_2

$$\underline{\text{Check}} \quad \frac{7}{2} \begin{bmatrix} 2 \\ -4 \end{bmatrix} + (-1) \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \end{bmatrix} \checkmark$$

Ex 2.3 //

$$A = \begin{bmatrix} 3 & -1 & 5 \\ -2 & 0 & -4 \\ -1 & 4 & 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$$

\vec{b} is in the span of the columns of A if and only if \vec{b} can be written as a linear combination of the column vectors from A ; i.e. there is at least one ordered triple solution to

$$x_1 \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ h \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & h \\ -1 & 4 & 2 & 2 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ -2 & 0 & -4 & h \\ 3 & -1 & 5 & 3 \end{array} \right]$$

$$\begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ 3R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & h-4 \\ 0 & 11 & 11 & 9 \end{array} \right]$$

$$\frac{11}{8}R_2 + R_3 \rightarrow R_3 \left[\begin{array}{ccc|c} -1 & 4 & 2 & 2 \\ 0 & -8 & -8 & h-4 \\ 0 & 0 & 0 & \frac{11}{8}h + \frac{7}{2} \end{array} \right]$$

$\therefore \vec{b}$ is in the span of the column vectors of A if and only if

$$\frac{11}{8}h + \frac{7}{2} = 0 \text{ ; i.e. } h = -28/11$$

Cx //

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & -28/11 \\ -1 & 4 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 114/11 \\ 0 & 1 & 1 & 9/11 \\ 0 & 0 & 0 & 0 \end{array} \right] \leftarrow \text{no contradiction}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 3 \\ -2 & 0 & -4 & 7 \\ -1 & 4 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \leftarrow \text{contradiction}$$

Ex 2.4

Dimensional Analysis

 $A\vec{u}$ $2 \times 3 \quad 2 \times 1$

non-match

 $A\vec{u}$ does not exist $A\vec{v}$ $2 \times 3 \quad 3 \times 1$

match

 $A\vec{v}$ is defined and its dimension are 2×1 $B\vec{u}$ $3 \times 2 \quad 2 \times 1$

match

 $B\vec{u}$ is 3×1 $B\vec{v}$ $3 \times 2 \quad 3 \times 1$

non-match

no $B\vec{v}$

$$A\vec{v} = \begin{bmatrix} 2 & -6 & -1 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$$

$$= (-2) \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (4) \begin{bmatrix} -6 \\ 5 \end{bmatrix} + (8) \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(2) + (4)(-6) + (8)(-1) \\ (-2)(0) + (4)(5) + (8)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -36 \\ 36 \end{bmatrix}$$

$$B\vec{u} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$= (10) \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 46 \\ -6 \end{bmatrix}$$

Ex 2.5

$$\begin{bmatrix} -1 & 2 & 5 \\ 8 & -2 & 0 \\ 1 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ -3 \end{bmatrix}$$

Matrix equation

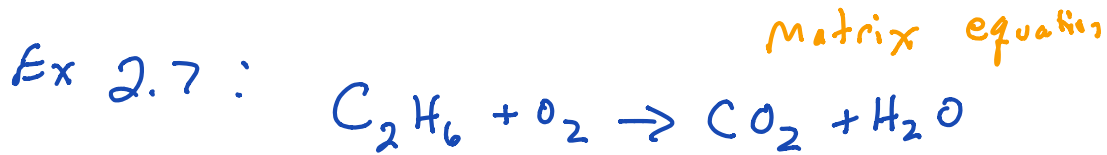
vector equation.

Ex 2.6

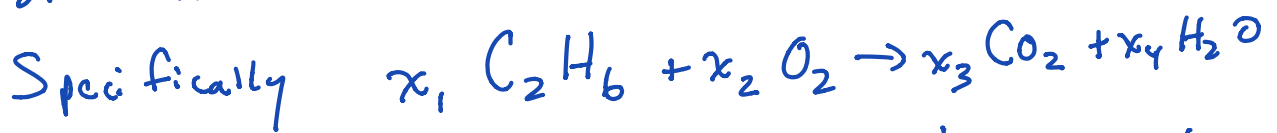
$$\begin{cases} 2x_1 - 3x_2 + x_3 - 2x_4 = 0 \\ 5x_1 - x_2 + x_4 = 9 \end{cases} \Rightarrow x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

vector equation

$$\text{System of equations} \Rightarrow \begin{bmatrix} 2 & -3 & 1 & -2 \\ 5 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$



Let's let x_1, \dots, x_4 represent the number of molecules in a balanced equation.



Let's represent each molecule by the vector

$$\begin{bmatrix} \# \text{ of carbon atoms} \\ \# \text{ of oxygen atoms} \\ \# \text{ of hydrogen atoms} \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

our system is:
$$\begin{cases} 2x_1 - x_3 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \\ 6x_1 - 2x_4 = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 6 & 0 & 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -7/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{bmatrix}$$

general solution:
$$\begin{cases} x_1 = \frac{1}{3}x_4 \\ x_2 = \frac{7}{6}x_4 \\ x_3 = \frac{2}{3}x_4 \\ x_4 \text{ is free} \end{cases}$$

you can't have fractional molecules, so I choose
to let $x_4 = 6$.

\therefore A balanced equation is $2C_2H_6 + 7O_2 \rightarrow 4CO_2 + 6H_2O$

Check: H: $2(6) = 6(2)$ ✓

O: $7(2) = 4(2) + 6(1)$ ✓

C: $2(2) = 4(1)$ ✓