Ex. 1.1 $\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases}$

Elimination Method

$$\begin{cases} 2x_1 - 3x_2 = 8 \\ 6x_1 + x_2 = -36 \end{cases} \xrightarrow{-3E_1 + E_2 \to E_2} \begin{cases} 2x_1 - 3x_2 = 8 \\ 10x_2 = -60 \end{cases}$$

$$\xrightarrow{\frac{1}{10}E_2 \to E_2} \begin{cases} 2x_1 - 3x_2 = 8 \\ x_2 = -6 \end{cases}$$

$$\xrightarrow{3E_2 + E_1 \to E_1} \begin{cases} 2x_1 = -10 \\ x_2 = -6 \end{cases}$$

$$\xrightarrow{\frac{1}{2}E_1 \to E_1} \begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$$

$\therefore$ The solution is $(-5, -6)$

(or you could write $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$

Check: E_1: $2(-5) - 3(-6) = 8$ ✓

E_2: $6(-5) + (-6) = -36$ ✓

Gauss-Jordan row elimination is the elimination method without writing down redundant information.

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 6 & 1 & -36 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 10 & -60 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{10}R_2 \to R_2} \left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{3R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 2 & 0 & -10 \\ 0 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -6 \end{array}\right]$$

$\therefore$ The solution is $\begin{cases} x_1 = -5 \\ x_2 = -6 \end{cases}$

Example 1.2

a. The given system is represented in augmented matrix form as:

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right]$$

Zeros below the leading entries created left-to-right.

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -23 \\ -5 & 1 & -2 & -7 \\ 1 & 3 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ \boxed{-5} & 1 & -2 & -7 \\ \boxed{2} & -5 & -3 & -23 \end{array}\right]$$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{5R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ \boxed{0} & 16 & 3 & 8 \\ \boxed{0} & \boxed{-11} & -5 & -29 \end{array}\right]$$

$$\frac{11}{16}R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 16 & 3 & 8 \\ 0 & 0 & -47/16 & -47/2 \end{array}\right]$$

A System equivalent to the given System is :

$$\begin{cases} x_1 + 3x_2 + x_3 = 3 \\ 16x_2 + 3x_3 = 8 \\ -\frac{47}{16}x_3 = -\frac{47}{2} \end{cases}$$

Solve bottom-up using back substitution

E_3 : $-\frac{47}{16}x_3 = -\frac{47}{2} \Rightarrow x_3 = 8$

E_2 : $16x_2 + 3(8) = 8 \Rightarrow x_2 = -1$

E_1 $x_1 + 3(-1) + (8) = 3 \Rightarrow x_1 = -2$

check, check, Check

The Solution is $\begin{cases} x_1 = -2 \\ x_2 = -1 \\ x_3 = 8 \end{cases}$

b.

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ -2 & 3 & -2 & 2 & -3 \\ 0 & -1 & -2 & 0 & -11 \\ 5 & -10 & 0 & -3 & -1 \end{array}\right] \begin{array}{c} 2R_1 + R_2 \to R_2 \\ -5R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$$

$$-R_2 + R_3 \to R_3 \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & -8 & -21 \end{array}\right]$$

$$-2R_3 + R_4 \to R_4 \quad \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & -1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

The last equation in the new (equivalent) system is $0 = 11$, this is called a contradiction. The solution set to the original system is $\emptyset$.

$$A = 7.3 \left[\begin{array}{ccc|c} 2 & 5 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot Columns: For R1, C1

For R2, C3

R3 Has no pivot column.

This matrix implies three variables but only has two pivot columns.

The right most column is <u>not</u> a pivot column (So no contradiction)

Situation iii, the system has "an infinite number" of solutions

$$B = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right]$$ Pivot Columns. C1, C2

Two variables

Situation (i) : exactly one solution. $(7, 2)$

$$C = \left[\begin{array}{cccc|c} -2 & 1 & -1 & 6 & 0 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns: C1, C2, C5

4 variables, 3 pivot columns, right-most column is a pivot column (contradiction)

The equation $0=6$ is a contradiction, So the system is inconsistent. (situation ii)

Row Echelon Form vs. Reduced Row Echelon Form.

your text	every other text & calculator
echelon form	row echelon form (REF)
reduced echelon form	reduced row echelon form (RREF)

Ex. 1.4 a) $\begin{cases} 2x_1 + 3x_2 = -2 \\ 6x_1 - 6x_2 = -1 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 6 & -6 & -1 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & -15 & 5 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & -15 & 5 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -\frac{1}{15}R_2 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & -1/3 \end{array}\right]$$

$\therefore$ The solution is $\begin{cases} x_1 = -1/2 \\ x_2 = -1/3 \end{cases}$

Check: E_1 : $2(-1/2) + 3(-1/3) = -2$ ✓

E_2 $6(-1/2) - 6(-1/3) = -1$ ✓

b.

$$\left[\begin{array}{ccc|c} 0 & 2 & -6 & -2 \\ 4 & -1 & 3 & 1 \\ -1 & 3 & -8 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 4 & -1 & 3 & 1 \\ 0 & 2 & -6 & -2 \end{array}\right]$$

$4R_1 + R_2 \to R_2$

$\frac{1}{2}R_3 \to R_3$

$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 11 & -29 & -15 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} -1 & 3 & -8 & -4 \\ 0 & 1 & -3 & -1 \\ 0 & 11 & -29 & -15 \end{array}\right]$$

$-11R_2 + R_3 \to R_3$

$-1R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 8 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

$\frac{1}{4}R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -3 & 8 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$-8R_3 + R_1 \to R_1$

$3R_3 + R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$3R_2 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$

By golly, the solution to the given system is $\begin{cases} x_1 = 0 \\ x_2 = -4 \\ x_3 = -1 \end{cases}$ ✓, ✓, ✓ (for reals!)

PHW #6

$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -2 & 4 & 3 & -7 \\ 6 & -12 & -7 & 10 \end{array}\right]$ $\begin{array}{l} 2R_1 + R_2 \to R_2 \\ -6R_1 + R_3 \to R_3 \end{array}$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -13 & -2 \end{array}\right]$

$\frac{13}{5}R_2 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & -49/5 \end{array}\right]$

$-\frac{5}{49}R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\begin{array}{l} -2R_3 + R_1 \to R_1 \\ 3R_3 + R_2 \to R_2 \end{array}$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\frac{1}{5}R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$-R_2 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\therefore$ Since the new third equation is $0=1$, the given system has no solution.

1.4c)

$$\left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 3 & -2 & 1 & 5 & -1 \\ 2 & 4 & -1 & 5 & 8 \\ -3 & -1 & 1 & -7 & -6 \end{array}\right] \quad \begin{array}{l} 3R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \\ -3R_1+R_4 \to R_4 \end{array} \quad \left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & -19 & 7 & -7 & -33 \end{array}\right]$$

$$\begin{array}{l} -1R_2+R_3 \to R_3 \\ \frac{19}{16}R_2+R_4 \to R_4 \end{array} \quad \left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17/16 & -17/16 & -17/8 \end{array}\right]$$

$$\frac{16}{17}R_4 \to R_4 \quad \left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

$$R_3 \leftrightarrow R_4 \quad \left[\begin{array}{cccc|c} -1 & 6 & -2 & 0 & 9 \\ 0 & 16 & -5 & 5 & 26 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} 2R_3+R_1 \to R_1 \\ 5R_3+R_2 \to R_2 \end{array} \quad \left[\begin{array}{cccc|c} -1 & 6 & 0 & -2 & 5 \\ 0 & 16 & 0 & 0 & 16 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{16}R_2 \to R_2 \\ -1R_1 \to R_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & -6 & 0 & 2 & -5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$6R_2+R_1 \to R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ A system equivalent to the given system

is: $\begin{cases} x_1 \qquad\qquad + 2x_4 = 1 \\ \quad x_2 \qquad\qquad\quad = 1 \\ \qquad\quad x_3 - x_4 = -2 \\ \qquad\qquad\qquad 0 = 0 \end{cases}$

This equivalent system is a <u>dependent</u> system of equations (fewer $\overset{\text{non-zero}}{\wedge}$ eqs < unknowns, no contradictions).

The <u>general</u> solution is:

$$\begin{cases} x_1 = -2x_4 + 1 \\ x_2 = 1 \\ x_3 = x_4 - 2 \\ x_4 \text{ is a free variable} \end{cases}$$

[note: x_1 & x_3 are dependent variables (dependent upon the value chosen for x_4). x_2 is a fixed variable]

Two specific solutions are: $\begin{cases} x_1 = -3 \\ x_2 = 1 \\ x_3 = 0 \\ x_4 = 2 \end{cases}$ and $\begin{cases} x_1 = 15 \\ x_2 = 1 \\ x_3 = -9 \\ x_4 = -7 \end{cases}$

Check the general solution

Eq 1: $-(-2x_4 + 1) + 6(1) - 2(x_4 - 2)$

$= 2x_4 - 1 + 6 - 2x_4 + 4$

$= 9$ ✓

Eq 2: $3(-2x_4 + 1) - 2(1) + (x_4 - 2) + 5(x_4)$

$= -6x_4 + 3 - 2 + x_4 - 2 + 5x_4$

$= -1$ ✓

Eq 3: $2(-2x_4+1)+4(1)-(x_4-2)+5(x_4)$

$= -4x_4+2+4-x_4+2+5x_4$

$= 8$ ✓

Eq 4: $-3(-2x_4+1)-(1)+(x_4-2)-7(x_4)$

$= 6x_4-3-1+x_4-2-7x_4$

$= -6$ ✓

Ex: 1.5 a. $\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 = 2 \\ x_2 - x_3 + 2x_4 = -3 \end{cases}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right] \xrightarrow{2R_2+R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right]$$

A system equivalent to the given system is:

$$\begin{cases} x_1 + x_4 = -4 \\ x_2 - x_3 + 2x_4 = -3 \end{cases}$$

$\therefore$ The general solution is:

$$\begin{cases} x_1 = -x_4 - 4 \\ x_2 = x_3 - 2x_4 - 3 \\ x_3 \text{ is a free variable} \\ x_4 \text{ is a free variable} \end{cases}$$

and two specific solutions are

$$\begin{cases} x_1 = -4 \\ x_2 = -2 \\ x_3 = 1 \\ x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = -5 \\ x_2 = -5 \\ x_3 = 0 \\ x_4 = 1 \end{cases}$$

1.5 b) $\begin{cases} 2x_1 + 6x_2 + 5x_3 = -2 \\ -x_1 - 3x_2 + 3x_3 = 1 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 6 & 5 & -2 \\ -1 & -3 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The general solution is:

$$\begin{cases} x_1 = -3x_2 - 1 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$$

Two specific solutions are

$$\begin{cases} x_1 = -4 \\ x_2 = 1 \\ x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = -7 \\ x_2 = 2 \\ x_3 = 0 \end{cases}$$

Ck//

$2(-4) + 6(1) + 5(0) = -2$ ✓ $\qquad 2(-7) + 6(2) + 5(0) = -2$ ✓

$-(-4) - 3(1) + 3(0) = 1$ ✓ $\qquad -(-7) - 3(2) + 3(0) = 1$ ✓

Ex. 1.6: $\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + 3x_2 - x_3 = b \\ -2x_1 - 8x_2 + 5x_3 = c \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 3 & -1 & b \\ -2 & -8 & 5 & c \end{array}\right] \xrightarrow[2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & -6 & 9 & 2a+c \end{array}\right]$$

$$\xrightarrow{3R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & -3 & -a+b \\ 0 & 0 & 0 & -a+3b+c \end{array}\right]$$

$\therefore$ This system is consistent if and only if $-a+3b+c=0$; i.e. $c = a-3b$

Check:

① $a=1,\ b=1,\ c=-2$

② $a=1,\ b=1,\ c=0$ ← not -2

① $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & -1 & 1 \\ -2 & -8 & 5 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 1 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ← no contradiction!

② $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & -1 & 1 \\ -2 & -8 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ ← contradiction!

Checked!

Ex. 1.7

Table 1:	Flow rate in = Flow rate out
A	$25 + 50 = x_1 + x_4$
B	$x_1 = x_2 + 50$
C	$x_2 + x_3 + 25 = 150$
D	$x_4 + 100 = x_3$

Flow rate system

$$\begin{cases} x_1 + x_4 = 75 \\ x_1 - x_2 = 50 \\ x_2 + x_3 = 125 \\ x_3 - x_4 = 100 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 1 & -1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 125 \\ 0 & 0 & 1 & -1 & 100 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 75 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

General Solution: $\begin{cases} x_1 = -x_4 + 75 \\ x_2 = -x_4 + 25 \\ x_3 = x_4 + 100 \\ x_4 \text{ is free} \end{cases}$

Flow rates cannot be negative.

$\therefore \quad 0 \le x_4 \le 25$ (x_2 cannot be negative)

$\therefore$
$$50 \le x_1 \le 75$$
$$0 \le x_2 \le 25$$
$$100 \le x_3 \le 125$$
$$0 \le x_4 \le 25$$